Problem 11922
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Find every positive integer n such that both n and n^2 are palindromes when written in the binary numeral system (and with no leading zeros).

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There are only two positive integers n such that both n and n^2 are palindromes in base 2: 1, and $3 = 11_2$. Note that in any base $b > 2$ there are infinite palindromes whose squares are palindromes too: for any $k \geq 0$, if $n := 10^k1_b$ then $n^2 = 10^k20^k1_b$ where 0^k means a string of k zeros.

We first state and prove the following general fact. If $n \equiv a$ modulo 2^m then $n^2 = (a + q2^m)^2 = a^2 + 2a2^m + 2^{2m} \equiv a^2$ modulo 2^{m+1}. Hence the m least significant bits of the binary representation of n determine exactly the $m + 1$ least significant bits of the binary representation of n^2.

Let n be a binary palindrome. We divide the proof in several cases with respect to d i. e. the number of 1s in the binary representation of n.

1) If $d = 1$ then $n = 1 = 1_2$ whose square is trivially a binary palindrome.

2) If $d = 2$ then $n = 2^k + 1 = 10^{k-1}1_2$ with $k \geq 1$. If $k = 1$ then $3 = 11_2$ whose square $3^2 = 1001_2$ is a binary palindrome. If $k \geq 2$ then $2k > k + 1$ and

$$n^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1_2$$

is not a binary palindrome.

3) If $d = 3$ then $n = 2^{2k} + 2^k + 1 = 10^{k-1}10^{k-1}1_2$ with $k \geq 1$. If $k = 1$ then $n = 7$ whose square is not a binary palindrome. If $k \geq 2$ then $4k > 3k + 1 > 2k + 1 > 2k > k + 1 > 0$, and

$$n^2 = 2^{4k} + 2^{3k+1} + 2^{2k+1} + 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^{k-1}110^{k-2}10^k1_2$$

is not a binary palindrome.

4) Let $d \geq 4$ then $n = 2^j + 2^{j-k} + \cdots + 2^k + 1 = 10^{k-1}1 * 10^{k-1}1_2$ with $j - k > k \geq 1$.

4.1) If $d \geq 4$ and $k \geq 2$ then $j \geq 5$ and $2^j + 2^{j-k} < n < 2^j + 2^{j-k+1}$ implies

$$2^{2j} + 2^{2j+1-k} < n^2 < 2^{2j} + 2^{2j-k+2} + 2^{2j-2k+2} < 2^{2j+2}.$$

Since $n = *10^{k-1}1_2$, by the fact stated above, it follows that $n^2 = *10^k1_2$ and if n^2 is a binary palindrome then $n^2 = 10^k1*_2$. Thus $n^2 < 2^{2j} + 2^{2j-k}$ or $n^2 > 2^{2j+1} + 2^{2j-k}$ which contradict the previous inequalities.

4.2) If $d = 4$ and $k = 1$ then $n = 2^j + 2^{j-1} + 2 + 1 = 110^{j-3}11_2$ with $j \geq 3$. For $j = 3, 4, 5$ we have respectively $n = 15$, $n = 27$ and $n = 51$ whose squares are not binary palindromes. For $j \geq 6$,

$$n^2 = 2^{2j+1} + 2^{2j-2} + 2^{j+3} + 2^j + 9 = 10010^{j-6}10010^{j-4}1001$$

which is not a binary palindrome.

4.3) If $d = 5$ and $k = 1$ then $n = 2^{2j} + 2^{2j-1} + 2^j + 2 + 1 = 110^{j-2}10^{j-2}11_2$ with $j \geq 2$. For $j = 2, 3$ we have respectively $n = 31$, and $n = 107$ whose squares are not binary palindrome. For $j \geq 4$,

$$n^2 = 2^{4j+1} + 2^{4j-2} + 2^{3j+1} + 2^{3j} + 2^{2j+3} + 2^{2j+1} + 2^{j+2} + 2^{j+1} + 9 = 10010^{j-4}110^{j-4}1010^{j-2}110^{j-3}1001$$

which is not a binary palindrome.

4.4) Let $d \geq 6$ and $k = 1$ then $n = 2^j + 2^{j-1} + 2^{j-i} + \ldots + 2^i + 2 + 1 = 110^{i-2}1 * 10^{i-2}11_2$ with $j - i > i \geq 2$.

4.4.1) If $d \geq 6$ and $i = 2$ then $n = 111 * 111_2$ with $j \geq 5$. Now $7 \cdot 2^{j-2} < n < 2^{j+1}$ implies

$$2^{2j+1} < 49 \cdot 2^{2j-4} < n^2 < 2^{2j+2}.$$

Since $n^2 = *0001_2$, it follows that if n^2 is a binary palindrome then $n^2 = 1000*_2$ and

$$n^2 < 2^{2j+1} + 2^{2j-2} < \left(1 + 2^{-1} + 2^{-2}\right)^2 2^{2j}$$

which implies $n < 2^j + 2^{j-1} + 2^{j-2}$ that is against the fact that $n > 2^j + 2^{j-1} + 2^{j-2}$.

4.4.2) If $d \geq 6$, $i \geq 3$ then $n = 110^{i-2}1 * 10^{i-2}11_2$ with $j > 2i$. Now $3 \cdot 2^{j-1} < n < 2^{j+1}$ implies

$$2^{2j+1} < 9 \cdot 2^{2j-2} < n^2 < 2^{2j+2}.$$

Since $n^2 = *0^{i-3}1001_2$ it follows that if n^2 is a binary palindrome then $n^2 = 10010^{i-3}*_2$ and

$$n^2 < 2^{2j+1} + 2^{2j-2} + 2^{2j+1-i} = \left(2 + 2^{-2} + 2^{1-i}\right) 2^{2j} < \left(1 + 2^{-1} + 2^{-i}\right)^2 2^{2j}$$

which implies $n < 2^j + 2^{j-1} + 2^{j-i}$ that is against the fact that $n > 2^j + 2^{j-1} + 2^{j-i}$. $\square$