Problem 11921
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Prove

$$\log^2(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)2^{k+1}}+\log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k}+\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^k}=\frac{\zeta(4)+\log^4(2)}{4}$$

where $H_k=\sum_{j=1}^{k}1/j$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. Since

$$[z^{k+1}]\log^2(1-z)=[z^{k+1}]\left(-\sum_{j=1}^{\infty}\frac{z^j}{j}\right)^2=\sum_{j=1}^{k}\frac{1}{j(k+1-j)}=\frac{1}{k+1}\sum_{j=1}^{k}\left(\frac{1}{j}+\frac{1}{k+1-j}\right)=\frac{2H_k}{k+1},$$

we have that for $0<z<1$, $\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{k+1}=\frac{1}{2}\log^2(1-z)$. Then it suffices to show

$$\log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k}+\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^k}=\frac{\zeta(4)-\log^4(2)}{4}. \tag{1}$$

Now, for $0<z\leq 1$,

$$F(z):=\frac{1}{2}\int_0^z\frac{\log^2(1-t)}{t}\,dt=\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{(k+1)^2},\quad\text{and}\quad\int_0^z\frac{F(t)}{t}\,dt=\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{(k+1)^3},$$

and, by integrating by parts,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{(k+1)^3}&=\int_0^z F(t)\,d(\log(t))=[F(t)\log(t)]_{0^+}^{z}-\int_0^z F'(t)\log(t)\,dt\\&=\log(z)\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{(k+1)^2}-\frac{1}{2}\int_0^z\frac{\log(t)\log^2(1-t)}{t}\,dt.\end{aligned}$$

Therefore, for $0<z\leq 1$,

$$-2\log(z)\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{(k+1)^2}+2\sum_{k=1}^{\infty}\frac{H_k z^{k+1}}{(k+1)^3}=-\int_0^z\frac{\log(t)\log^2(1-t)}{t}\,dt \tag{2}$$

and, by letting $z=\frac{1}{2}$, we have that (1) is equivalent to

$$-\int_0^{\frac{1}{2}}\frac{\log(t)\log^2(1-t)}{t}\,dt=\frac{\zeta(4)-\log^4(2)}{4}. \tag{3}$$

We notice that

$$\begin{aligned}\int_0^{\frac{1}{2}}\frac{\log(t)\log^2(1-t)}{t}\,dt&=\frac{1}{2}\int_0^{\frac{1}{2}}\log^2(1-t)\,d(\log^2(t))\\&=\frac{1}{2}\left[\log^2(1-t)\log^2(t)\right]_{0^+}^{\frac{1}{2}}+\int_0^{\frac{1}{2}}\frac{\log(1-t)\log^2(t)}{1-t}\,dt\\&=\frac{\log^4(2)}{2}+\int_{\frac{1}{2}}^{1}\frac{\log(t)\log^2(1-t)}{t}\,dt.\end{aligned}$$

Hence, by using (2) for $z = 1$, we get

$$\begin{aligned}
-\int_0^{\frac{1}{2}} \frac{\log(t)\log^2(1-t)}{t}\,dt &= -\frac{1}{2}\int_0^{\frac{1}{2}} \frac{\log(t)\log^2(1-t)}{t}\,dt - \frac{1}{2}\left(\frac{\log^4(2)}{2} + \int_{\frac{1}{2}}^1 \frac{\log(t)\log^2(1-t)}{t}\,dt\right) \\
&= -\frac{\log^4(2)}{4} - \frac{1}{2}\int_0^1 \frac{\log(t)\log^2(1-t)}{t}\,dt \\
&= -\frac{\log^4(2)}{4} + \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3} = \frac{\zeta(4) - \log^4(2)}{4}
\end{aligned}$$

and the proof of (3) is complete because $\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3} = \zeta(3,1) = \zeta(4)/4$ which is a known result (see for example *Special Values of Multidimensional Polylogarithms*, Trans. Amer. Math. Soc. 353, 907-941, 2001 by Borwein, Bradley, Broadhurst, and Lisonek). $\square$