Problem 11920
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Proposed by A. Plaza and S. Falcon (Spain).

For positive integer k, let $\{F_{k,n}\}_{n\geq 0}$ be the sequence defined by initial conditions $F_{k,0}=0$, $F_{k,1}=1$, and the recurrence $F_{k,n+1}=kF_{k,n}+F_{k,n-1}$. Find a closed form for $\sum_{i=0}^{n}\binom{2n+1}{i}F_{k,2n+1-2i}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. By solving the linear recurrence with the given initial conditions, we obtain

$$F_{k,n}=\frac{\alpha_k^n-\beta_k^n}{\sqrt{k^2+4}} \qquad \text{with} \qquad \alpha_k=\frac{k+\sqrt{k^2+4}}{2} \qquad \text{and} \qquad \beta_k=\frac{k-\sqrt{k^2+4}}{2}.$$

Notice that α_k and β_k are the roots of the equation $z^2-kz-1=0$ ($F_{1,n}$ is the nth Fibonacci number). Hence, since $\alpha_k\cdot\beta_k=-1$, it follows that, for $n\geq 0$,

$$\begin{aligned}
\sum_{i=0}^{n}\binom{2n+1}{i}F_{k,2n+1-2i} &= \frac{1}{\sqrt{k^2+4}}\left(\alpha_k^{2n+1}\sum_{i=0}^{n}\binom{2n+1}{i}\alpha_k^{-2i}-\beta_k^{2n+1}\sum_{i=0}^{n}\binom{2n+1}{i}\beta_k^{-2i}\right)\\
&= \frac{1}{\sqrt{k^2+4}}\left(\alpha_k^{2n+1}\sum_{i=0}^{n}\binom{2n+1}{i}\alpha_k^{-2i}+\alpha_k^{-(2n+1)}\sum_{i=0}^{n}\binom{2n+1}{2n+1-i}\alpha_k^{2i}\right)\\
&= \frac{1}{\sqrt{k^2+4}}\left(\alpha_k^{2n+1}\sum_{i=0}^{n}\binom{2n+1}{i}\alpha_k^{-2i}+\alpha_k^{-(2n+1)}\sum_{j=n+1}^{2n+1}\binom{2n+1}{j}\alpha_k^{2(2n+1-j)}\right)\\
&= \frac{1}{\sqrt{k^2+4}}\left(\alpha_k^{2n+1}\sum_{i=0}^{n}\binom{2n+1}{i}\alpha_k^{-2i}+\alpha_k^{2n+1}\sum_{j=n+1}^{2n+1}\binom{2n+1}{j}\alpha_k^{-2j}\right)\\
&= \frac{\alpha_k^{2n+1}(1+\alpha_k^{-2})^{2n+1}}{\sqrt{k^2+4}}=\frac{(\alpha_k-\beta_k)^{2n+1}}{\sqrt{k^2+4}}=\frac{(\sqrt{k^2+4})^{2n+1}}{\sqrt{k^2+4}}=(k^2+4)^n.
\end{aligned}$$

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