Problem 11919
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Proposed by A. Alt (USA).

For positive integers m, n and k with $k \geq 2$, prove

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \left(\min\{i_1, \ldots, i_k\}\right)^m = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) \sum_{j=1}^{n} j^{k+m-i}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. In the set $\{1, 2, \ldots, n\}^k$ there are $\binom{k}{i}(n-j)^{k-i}$ points whose minimal coordinate value j appears i times. Hence

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \left(\min\{i_1, \ldots, i_k\}\right)^m = \sum_{j=1}^{n} j^m \sum_{i=1}^{k} \binom{k}{i} (n-j)^{k-i}.$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} \left((n+1)^i - n^i\right) \sum_{j=1}^{n} j^{k+m-i} &= \sum_{j=1}^{n} j^k \sum_{i=0}^{m} \binom{m}{i} \left((n+1)^i - n^i\right) (-j)^{m-i} \\
&= \sum_{j=1}^{n} j^k \left((n+1-j)^m - (n-j)^m\right) \\
&= \sum_{j=1}^{n} j^k \left(n-(j-1)\right)^m - \sum_{j=0}^{n-1} j^k (n-j)^m \\
&= \sum_{j=0}^{n-1} (j+1)^k (n-j)^m - \sum_{j=0}^{n-1} j^k (n-j)^m \\
&= \sum_{j=0}^{n-1} (n-j)^m \left((j+1)^k - j^k\right) \\
&= \sum_{j=0}^{n-1} (n-j)^m \sum_{i=1}^{k} \binom{k}{i} j^{k-i} \\
&= \sum_{j=1}^{n} j^m \sum_{i=1}^{k} \binom{k}{i} (n-j)^{k-i}.
\end{aligned}$$

□