Problem 11918
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Proposed by Le Van Phu Cuong (Vietnam).

Let f be n times continuously differentiable on $[0,1]$, with $f(1/2)=0$ and $f^{(i)}(1/2)=0$ when i is even and less than n. Prove

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{(2n+1)4^n(n!)^2}\int_0^1 \left(f^{(n)}(x)\right)^2\,dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. If $g \in C^n([0,1])$, then by integrating by parts we obtain

$$\int_0^a g(x)\,dx = \sum_{i=0}^{n-1}\frac{(-1)^i a^{i+1} g^{(i)}(a)}{(i+1)!} + \frac{(-1)^n}{n!}\int_0^a x^n g^{(n)}(x)\,dx$$

for any $a \in (0,1)$. Hence

$$\int_0^{1/2} f(x)\,dx = \sum_{i=0}^{n-1}\frac{(-1)^i f^{(i)}(1/2)}{2^{i+1}(i+1)!} + \frac{(-1)^n}{n!}\int_0^{1/2} x^n f^{(n)}(x)\,dx,$$

and

$$\int_{1/2}^1 f(x)\,dx = \int_0^{1/2} f(1-x)\,dx = \sum_{i=0}^{n-1}\frac{f^{(i)}(1/2)}{2^{i+1}(i+1)!} + \frac{1}{n!}\int_0^{1/2} x^n f^{(n)}(1-x)\,dx.$$

Since $f^{(i)}(1/2)=0$ when i is even and less than n, it follows that

$$\int_0^1 f(x)\,dx = \int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx = \frac{1}{n!}\left((-1)^n\int_0^{1/2} x^n f^{(n)}(x)\,dx + \int_0^{1/2} x^n f^{(n)}(1-x)\,dx\right).$$

Finally, by Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\int_0^1 f(x)\,dx\right)^2 &\leq \frac{2}{(n!)^2}\left(\left(\int_0^{1/2} x^n f^{(n)}(x)\,dx\right)^2 + \left(\int_0^{1/2} x^n f^{(n)}(1-x)\,dx\right)^2\right)\\ &\leq \frac{2}{(n!)^2}\left(\int_0^{1/2} x^{2n}\,dx\int_0^{1/2}\left(f^{(n)}(x)\right)^2\,dx + \int_0^{1/2} x^{2n}\,dx\int_0^{1/2}\left(f^{(n)}(1-x)\right)^2\,dx\right)\\ &\leq \frac{1}{(2n+1)4^n(n!)^2}\int_0^1\left(f^{(n)}(x)\right)^2\,dx,\end{aligned}$$

where we used

$$\int_0^{1/2} x^{2n}\,dx = \frac{1}{(2n+1)2^{2n+1}} \quad \text{and} \quad \int_0^{1/2}\left(f^{(n)}(1-x)\right)^2\,dx = \int_{1/2}^1\left(f^{(n)}(x)\right)^2\,dx.$$

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