Problem 11917
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Let A be a 2×2 matrix with rational entries and both eigenvalues less than one in absolute value. Prove that

$$\log(I - A) := -\sum_{k=1}^{\infty} \frac{A^k}{k}$$

has rational entries if and only if $A^2 = 0$.

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Solution. Let $L = \log(I - A)$. If $A^2 = 0$ then $L = -A$ and therefore L has rational entries.
Now we assume that L has rational entries. It is known (see K. S. Williams, *The nth power of a 2×2 matrix*, Math. Mag. 65(1992), 336) that for $k \geq 1$,

$$A^k = \begin{cases} \alpha^k \left(\dfrac{A - \beta I}{\alpha - \beta}\right) - \beta^k \left(\dfrac{A - \alpha I}{\alpha - \beta}\right), & \text{if } \alpha \neq \beta \\ \\ \alpha^k I + k\alpha^{k-1}(A - \alpha I), & \text{if } \alpha = \beta. \end{cases}$$

where α and β are the eigenvalues of the matrix A.
Hence, since $|\alpha| < 1$ and $|\beta| < 1$, we have

$$L = \begin{cases} \log(1-\alpha) \left(\dfrac{A - \beta I}{\alpha - \beta}\right) - \log(1-\beta) \left(\dfrac{A - \alpha I}{\alpha - \beta}\right), & \text{if } \alpha \neq \beta \\ \\ \log(1-\alpha)I - \dfrac{A - \alpha I}{1 - \alpha}, & \text{if } \alpha = \beta. \end{cases}$$

If $\alpha = \beta$ then,

$$l_{11} = \log(1-\alpha) - \frac{a_{11} - \alpha}{1 - \alpha} \quad \Rightarrow \quad \log(1-\alpha) = l_{11} + \frac{a_{11} - \alpha}{1 - \alpha}$$

Since $l_{11}, a_{11} \in \mathbb{Q}$ and α is an algebraic number, it follows that $1 - \alpha$ and $l_{11} + \frac{a_{11}-\alpha}{1-\alpha}$ are algebraic numbers. By Hermite-Lindemann theorem, the natural logarithm of a positive algebraic number different from 1 is transcendental, therefore $1 - \alpha = 1$, that is $\alpha = 0$. Therefore $\text{tr}(A) = 2\alpha = 0$, $\det(A) = \alpha^2 = 0$ which imply, by Cayley-Hamilton theorem, that $A^2 = 0$.
Now, it suffices to show that the condition $\alpha \neq \beta$ leads always to a contradiction. If $a_{21} \neq 0$ then

$$(\alpha - \beta)l_{21} = \log(1-\alpha)a_{21} - \log(1-\beta)a_{21} \quad \Rightarrow \quad \log\left(\frac{1-\alpha}{1-\beta}\right) = \frac{(\alpha - \beta)l_{21}}{a_{21}}.$$

Hence, by using the same argument as above, we have that $(1-\alpha)/(1-\beta) = 1$, that is $\alpha = \beta$. Contradiction.
So $a_{21} = 0$ and, in a similar way, we get $a_{12} = 0$. Thus, $a_{11} = \alpha$, $a_{22} = \beta$ and

$$(\alpha - \beta)l_{11} = \log(1-\alpha)(a_{11} - \beta) - \log(1-\beta)(a_{11} - \alpha) = \log(1-\alpha)(\alpha - \beta) \ \Rightarrow \ \log(1-\alpha) = l_{11},$$
$$(\alpha - \beta)l_{22} = \log(1-\alpha)(a_{22} - \beta) - \log(1-\beta)(a_{22} - \alpha) = -\log(1-\beta)(\beta - \alpha) \ \Rightarrow \ \log(1-\beta) = l_{22}.$$

Therefore, again by using the same argument, we conclude that $\alpha = 0$ and $\beta = 0$. Contradiction. $\square$

Remark. If A is a 2×2 matrix with integer entries and both eigenvalues less than one in absolute value then it follows immediately that $\text{tr}(A) = \det(A) = 0$, and therefore $A^2 = 0$ by Cayley-Hamilton theorem.