Problem 11916
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Show that if n, r, and s are positive integers, then

$$\binom{n+r}{n}\sum_{k=0}^{s-1}\binom{r+k}{r-1}\binom{n+k}{n}=\binom{n+s}{n}\sum_{k=0}^{r-1}\binom{s+k}{s-1}\binom{n+k}{n}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We have to show that $T(s,r,x)=T(r,s,x)$ where

$$T(s,r,x)=\frac{1}{\binom{x+s}{s}}\sum_{k=0}^{s-1}\binom{r+k}{r-1}\binom{x+k}{k}=s!\sum_{k=0}^{s-1}\frac{\binom{r+k}{r-1}}{k!}\frac{1}{(x+s)\cdots(x+k+1)}.$$

Since $x\to T(s,r,x)$ is a rational functions with simple poles at $-1,-2,\dots,-s$, by partial-fraction decomposition it follows that there exist real numbers $A_1,A_2,\dots,A_s$ such that

$$T(s,r,x)=\sum_{i=1}^{s}\frac{A_i}{x+i}.$$

Then, for $1\leq j\leq s$,

$$\lim_{x\to -j}(x+j)T(s,r,x)=\lim_{x\to -j}(x+j)\sum_{i=1}^{s}\frac{A_i}{x+i}=A_j$$

and

$$\begin{aligned}
A_j&=\lim_{x\to -j}(x+j)T(s,r,x)\\
&=\lim_{x\to -j}s!\sum_{k=0}^{s-1}\frac{\binom{r+k}{r-1}}{k!}\frac{(x+j)}{(x+s)\cdots(x+k+1)}\\
&=s!\sum_{k=0}^{s-1}\frac{\binom{r+k}{r-1}}{k!}\frac{1}{(-j+s)\cdots 1\cdot(-1)\cdots(-j+k+1)}\\
&=s!\sum_{k=0}^{s-1}\frac{\binom{r+k}{r-1}}{k!}\frac{(-1)^{j-k-1}}{(s-j)!(j-k-1)!}\\
&=j\binom{s}{j}\sum_{k=0}^{j-1}(-1)^{j-1-k}\binom{j-1}{k}\binom{r+k}{r-1}=j\binom{s}{j}\binom{r}{j}
\end{aligned}$$

where in the last step we used the identity (3.48) in Gould's book *Combinatorial Identities* with $n=j-1$ and $m=r-1$,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{r+k}{m}=\binom{r}{m-n}.$$

Hence

$$T(s,r,x)=\sum_{j=1}^{\min(r,s)}\frac{j\binom{s}{j}\binom{r}{j}}{x+j}=T(r,s,x).$$

□