Problem 11915
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Given four points A, B, C, and D in order on a line in Euclidean space, under what conditions will there be a point P off the line such that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have equal measure?

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Solution. We will show that the point P exists if and only if $\dfrac{|AD||BC|}{|AB||CD|} < 3$.

In the complex plane, we assume without loss of generality that

$$A = 0,\ B = a,\ C = a + b,\ D = a + b + c \text{ with } a, b, c \text{ real positive numbers.}$$

We first note that if $a = b$ then the locus of points P such that $|\angle APB| = |\angle BPC|$ is the vertical line $\mathrm{Re}(z) = B$. On the other hand, if $a \neq b$, by using the Möbius transformation (which preserves angles),

$$M(z) = \frac{2bz}{(b - a)z + a(a + b)}$$

we have that $M(A) = 0$, $M(B) = 1$, $M(C) = 2$, so the locus is $M^{-1}(\{\mathrm{Re}(z) = 1\})$, which is the circle with a diameter of endpoints B and $B' = a(a + b)/(a - b)$ (the pole of M).
Similarly, the locus of points P such that $|\angle BPC| = |\angle CPD| =$ is the vertical line $\mathrm{Re}(z) = C$ when $b = c$ and the circle with a diameter of endpoints C and $C' = a + b(b + c)/(b - c)$ otherwise.
Hence it suffices to give a condition such that the two loci have non-empty intersection.
It is easy to see that if z_1, z_2, z_3, z_4 are four distinct points on the real line then the circle with a diameter of endpoints z_1, z_2 and the circle with a diameter of endpoints z_3, z_4 have non empty intersection if and only if the cross-ratio

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is negative.
Hence, in our case (we consider the line as a particular kind of circle with a point at infinity),

$$(B, B'; C, C') = \left(a, \frac{a(a + b)}{a - b}; a + b, a + \frac{b(b + c)}{b - c}\right) = \frac{b(a + b + c) - 3ac}{(a + b)(b + c)} < 0$$

that is $b(a + b + c) < 3ac$ or $\dfrac{|AD||BC|}{|AB||CD|} < 3$. $\square$