Problem 11914
(American Mathematical Monthly, Vol.123, May 2016)

Proposed by R. Chapman (UK) and R. Tauraso (Italy).

Show that for all positive integers m and n,

$$\sum_{k=1}^{n}(-4)^{-k}\binom{n-k}{k-1}\sum_{j=1}^{3m}(-2)^{-j}\binom{n+1-2k}{j-1}\binom{m-k}{3m-j}=0.$$

(Here, $\binom{x}{k}=\prod_{i=0}^{k-1}(x-i)/k!$ for $x\in\mathbb{R}$).

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

Let $S(m,n)$ be the left-hand side then

$$\begin{aligned}S(m,n)&=-\frac{1}{2}\sum_{k=1}^{n}(-4)^{-k}\binom{n-k}{k-1}[x^{3m-1}]\left((1-x/2)^{n+1-2k}(1+x)^{m-k}\right)\\&=-\frac{1}{2}\sum_{r+s=n-1,s\leq r}(-4)^{-1-s}\binom{r}{s}[x^{3m-1}]\left((1-x/2)^{r-s}(1+x)^{m-s-1}\right)\\&=\frac{1}{8}\sum_{r+s=n-1,s\leq r}(-1)^{s}2^{-r-s}\binom{r}{s}[x^{3m-1}]\left((2-x)^{r-s}(1+x)^{m-s-1}\right).\end{aligned}$$

For each positive integer m, introduce the generating function

$$F_m(y)=8\sum_{n=1}^{\infty}S(m,n)(2y)^n.$$

Thus

$$\begin{aligned}F_m(y)&=2y\sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\binom{r}{s}[x^{3m-1}]\left((2-x)^{r-s}(1+x)^{m-s-1}y^{r+s}\right)\\&=2y[x^{3m-1}](1+x)^{m-1}\sum_{r=0}^{\infty}(y(2-x))^r\sum_{s=0}^{r}\binom{r}{s}\left(\frac{-y}{(1+x)(2-x)}\right)^s\\&=2y[x^{3m-1}](1+x)^{m-1}\sum_{r=0}^{\infty}\left((2-x)y-\frac{y^2}{(1+x)}\right)^r\\&=2y[x^{3m-1}]\frac{(1+x)^{m-1}}{1-(2-x)y+y^2/(1+x)}\\&=2y[x^{3m-1}]\frac{(1+x)^m}{(1+x)-(2-x)(1+x)y+y^2}\\&=\frac{2y}{(1-y)^2}[x^{3m-1}]\frac{(1+x)^m}{1+ux+(u^2-u)x^2}\end{aligned}$$

where we define $u = 1/(1-y)$. Now

$$\frac{1}{1+ux+(u^2-u)x^2} = \frac{1}{(1-\alpha x)(1-\beta x)} = \sum_{k=0}^{\infty} a_k x^k$$

where

$$a_k = \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} \qquad \text{with} \qquad \alpha = \frac{-u+\sqrt{4u-3u^2}}{2} \quad , \quad \beta = \frac{-u-\sqrt{4u-3u^2}}{2}.$$

In order to prove that $F_m(y) = 0$, and therefore $S(m,n) = 0$, we need to show that for each positive integer m,

$$0 = \sum_{j=0}^{m} \binom{m}{j} a_{3m-1-(m-j)} = \sum_{j=0}^{m} \binom{m}{j} \frac{\alpha^{2m+j}-\beta^{2m+j}}{\alpha-\beta} = \frac{(\alpha^2(\alpha+1))^m-(\beta^2(\beta+1))^m}{\alpha-\beta}$$

which holds because it is straightforward to verify that

$$\alpha^2(\alpha+1) = u(\alpha+1)(\beta+1) = \beta^2(\beta+1).$$

$\square$