Problem 11910
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Proposed by C. I. Vălean (Romania).

Let F_k be the k-th Fibonacci number. Find

$$\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-3}}+\arctan\frac{1}{F_{4n-2}}+\arctan\frac{1}{F_{4n-1}}-\arctan\frac{1}{F_{4n}}\right).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\varphi=(\sqrt{5}+1)/2$, then for $n>0$,

$$\begin{aligned}\arctan(\varphi^{-(2n+1)})+\arctan(\varphi^{-(2n-1)})&=\arctan\left(\frac{\varphi^{-(2n+1)}+\varphi^{-(2n-1)}}{1-\varphi^{-4n}}\right)=\arctan\left(\frac{\varphi^{-1}+\varphi}{\varphi^{2n}-\varphi^{-2n}}\right)\\&=\arctan\left(\frac{\sqrt{5}}{\varphi^{2n}-\varphi^{-2n}}\right)=\arctan\frac{1}{F_{2n}}.\end{aligned}$$

Moreover, for $n>1$,

$$\begin{aligned}\arctan\frac{1}{F_{2n}}+\arctan\frac{1}{F_{2n-1}}&=\arctan\left(\frac{F_{2n}^{-1}+F_{2n-1}^{-1}}{1-F_{2n}^{-1}F_{2n-1}^{-1}}\right)=\arctan\left(\frac{F_{2n}+F_{2n-1}}{F_{2n}F_{2n-1}-1}\right)\\&=\arctan\left(\frac{F_{2n+1}}{F_{2n}F_{2n-1}-1}\right)=\arctan\frac{1}{F_{2n-2}}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=1}^{N}\left(\arctan\frac{1}{F_{4n-2}}-\arctan\frac{1}{F_{4n}}\right)&=\sum_{n=1}^{2N}(-1)^{n+1}\arctan\frac{1}{F_{2n}}\\&=\sum_{n=1}^{2N}(-1)^{n+1}\left(\arctan(\varphi^{-(2n+1)})+\arctan(\varphi^{-(2n-1)})\right)\\&=\sum_{n=2}^{2N+1}(-1)^{n}\arctan(\varphi^{-(2n-1)})+\sum_{n=1}^{2N}(-1)^{n+1}\arctan(\varphi^{-(2n-1)})\\&=\arctan(\varphi^{-1})-\arctan(\varphi^{-(4N+1)})\\&=\frac{\pi}{2}-\arctan(\varphi)-\arctan(\varphi^{-(4N+1)})\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{N}\left(\arctan\frac{1}{F_{4n-3}}+\arctan\frac{1}{F_{4n-1}}\right)&=\sum_{n=1}^{2N}\arctan\frac{1}{F_{2n-1}}\\&=\arctan\frac{1}{F_1}+\sum_{n=2}^{2N}\left(\arctan\frac{1}{F_{2n-2}}-\arctan\frac{1}{F_{2n}}\right)\\&=\arctan\frac{1}{F_1}+\arctan\frac{1}{F_2}-\arctan\frac{1}{F_{4N}}=\frac{\pi}{2}-\arctan\frac{1}{F_{4N}}.\end{aligned}$$

Finally, it follows that

$$\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-3}}+\arctan\frac{1}{F_{4n-2}}+\arctan\frac{1}{F_{4n-1}}-\arctan\frac{1}{F_{4n}}\right)=\pi-\arctan(\varphi).$$

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