Problem 11906
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Proposed by R. Bosch (USA).

Let x, y, and z be positive numbers such that $xyz = 1$. Prove that

$$\sqrt{\frac{x+1}{x^2-x+1}}+\sqrt{\frac{y+1}{y^2-y+1}}+\sqrt{\frac{z+1}{z^2-z+1}}\leq 3\sqrt{2}.$$

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Since $\sqrt{x}$ is concave in $[0,+\infty)$, it follows that

$$\sqrt{\frac{x+1}{x^2-x+1}}+\sqrt{\frac{y+1}{y^2-y+1}}+\sqrt{\frac{z+1}{z^2-z+1}}\leq 3\sqrt{\frac{(x+1)g(x)+(y+1)g(y)+(z+1)g(z)}{3}}$$
$$\leq 3\sqrt{\frac{3+g(x)+g(y)+g(z)}{3}}$$

where $g(t)=1/(1-t+t^2)$ satisfies $1-tg(t)=(1-t)^2/(1-t+t^2)\geq 0$.
Hence it suffices to show that if $x,y,z>0$ and $xyz=1$ then

$$g(x)+g(y)+g(z)\leq 3.$$

Finally, we notice that the following stronger inequality was proved in my solution of Problem 11867,

$$g(x)+g(y)+g(z)+(1-xg(x))(1-yg(y))(1-zg(z))\leq 3.$$

By expanding and by taking account of the constraint $xyz=1$, the above inequality is equivalent to

$$x^2y^2+y^2z^2+z^2x^2-3(xy+yz+zx)+6\geq 0,$$

or, by letting $X=1/x$, $Y=1/y$, $Z=1/z$, to

$$h(X,Y,Z):=X^2+Y^2+Z^2-3(X+Y+Z)+6\geq 0.$$

In order to prove it, we use the mixing variables method.
Without loss of generality we can assume that $Z\leq 1$, then $XY=1/Z\geq 1$ and

$$h(X,Y,Z)=h(\sqrt{XY},\sqrt{XY},Z)+(X-Y)^2-3(\sqrt{X}-\sqrt{Y})^2\geq h(\sqrt{XY},\sqrt{XY},Z)$$

because $(\sqrt{X}+\sqrt{Y})^2\geq 4\sqrt{XY}\geq 4$ and

$$(X-Y)^2-3(\sqrt{X}-\sqrt{Y})^2=(\sqrt{X}-\sqrt{Y})^2((\sqrt{X}+\sqrt{Y})^2-3)\geq 0.$$

Finally, $Z\leq 1$ implies

$$h(\sqrt{XY},\sqrt{XY},Z)=h(1/\sqrt{Z},1/\sqrt{Z},Z)=(Z^2+2Z\sqrt{Z}+2(1-\sqrt{Z}))(\sqrt{Z}-1)^2/Z\geq 0.$$

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