Problem 11905
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Proposed by C. Lupu (USA).

From a point P inside a triangle ABC, the perpendiculars PP_A, PP_B, and PP_C are drawn to its sides. Let R be the circumradius and r the inradius of the triangle. Prove that

$$\frac{R}{2r} \leq \frac{|PA||PB||PC|}{(|PP_B|+|PP_C|)(|PP_A|+|PP_C|)(|PP_A|+|PP_B|)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\frac{|PP_B|+|PP_C|}{|PA|} = \sin\alpha_1 + \sin\alpha_2$$

where $\alpha_1 = |P\widehat{A}C|$ and $\alpha_2 = |P\widehat{A}B|$.
Hence, by using similar notations for the other factors, the inquality is equivalent to

$$(\sin\alpha_1+\sin\alpha_2)(\sin\beta_1+\sin\beta_2)(\sin\gamma_1+\sin\gamma_2) \leq \frac{2r}{R} = 8\sin\left(\frac{\alpha_1+\alpha_2}{2}\right)\sin\left(\frac{\beta_1+\beta_2}{2}\right)\sin\left(\frac{\gamma_1+\gamma_2}{2}\right)$$

which holds because the function $\sin x$ is positive and concave in $(0,\pi)$ and

$$(\sin x_1 + \sin x_2) \leq 2\sin\left(\frac{x_1+x_2}{2}\right)$$

for all $x_1, x_2 \in (0,\pi)$. □