Problem 11903
(American Mathematical Monthly, Vol.123, April 2016)

Proposed by P. Perfetti (Italy).

Find a homogeneous polynomial P of degree two in a, b, c, and d such that $0 < a < b < c$ and $d > 0$,

$$\int_0^a \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx = \int_b^c \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx$$

if and only if $P(a,b,c,d) = \sqrt{d(a+d)(b+d)(c+d)}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

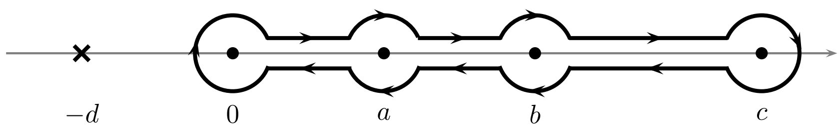
We will show that

$$P(a,b,c,d) = -\frac{a^2+b^2+c^2}{8} + d^2 + \frac{ab+ac+bc}{4} + \frac{(a+b+c)d}{2}.$$

Let

$$f(z) = \frac{\sqrt{z(a-z)(z-b)(z-c)}}{z+d}$$

which is, meromorphic on the entire complex plane $\mathbb{C}$ except the intervals $[0,a]$ and $[b,c]$ (here the square root $\sqrt{\cdot}$ is the branch on $\mathbb{C}$ minus the positive real axis with $\sqrt{-1} = i$).
Let γ be the clockwise closed curve γ given by



We have that

$$\int_\gamma f(z)dz = 2\int_0^a f(x)dx - 2\int_b^c f(x)dx.$$

Moreover, by the Residue Theorem

$$\int_\gamma f(z)dz = 2\pi i\left(\text{Res}(f,-d) + \text{Res}(f,\infty)\right).$$

Now

$$\text{Res}(f,-d) = \sqrt{-d(a+d)(-d-b)(-d-c)} = i\sqrt{d(a+d)(b+d)(c+d)}.$$

and

$$\begin{aligned}
\text{Res}(f,\infty) &= \text{Res}\left(-\frac{f(1/z)}{z^2},0\right) = \text{Res}\left(-\frac{f(1/z)}{z^2},0\right)\\
&= -i\text{Res}\left(\frac{\sqrt{(1-az)(1-bz)(1-cz)}}{z^3(1+dz)},0\right)\\
&= -i[z^2]\frac{\sqrt{(1-az)(1-bz)(1-cz)}}{(1+dz)}\\
&= -i[z^2]\left(1-\frac{az}{2}-\frac{a^2z^2}{8}\right)\left(1-\frac{bz}{2}-\frac{b^2z^2}{8}\right)\left(1-\frac{cz}{2}-\frac{c^2z^2}{8}\right)\left(1-dz+d^2z^2\right)\\
&= -iP(a,b,c,d).
\end{aligned}$$

Hence

$$\int_0^a f(x)\,dx - \int_b^c f(x)\,dx = \pi\left(-\sqrt{d(a+d)(b+d)(c+d)} + P(a,b,c,d)\right)$$

which implies that the two integrals on the left are equal if and only if

$$P(a,b,c,d) = \sqrt{d(a+d)(b+d)(c+d)}.$$

□