Problem 11899
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Proposed by J. Sorel (Romania).

Show that for any positive integer n,

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{k-1}=\binom{4n+1}{2n}+\binom{2n}{n}^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned}
\text{LHS} &= 1+\sum_{k=1}^{n}\binom{2n}{k}\left(\binom{2n}{k}+\binom{2n}{k-1}\right)+\sum_{k=n+1}^{2n}\left(\binom{2n}{k}+\binom{2n}{k-1}\right)\binom{2n}{k-1}+1\\
&=\sum_{k=0}^{n}\binom{2n}{k}^2+\sum_{k=1}^{2n}\binom{2n}{k}\binom{2n}{k-1}+\sum_{k=n+1}^{2n+1}\binom{2n}{k-1}^2\\
&=\sum_{k=0}^{n}\binom{2n}{k}^2+\sum_{k=1}^{2n}\binom{2n}{k}\binom{2n}{k-1}+\sum_{k=n}^{2n}\binom{2n}{k}^2\\
&=\sum_{k=0}^{2n}\binom{2n}{k}\binom{2n}{n-k}+\sum_{k=1}^{2n}\binom{2n}{2n-k}\binom{2n}{k-1}+\binom{2n}{n}^2\\
&=\binom{4n}{2n}+\binom{4n}{2n-1}+\binom{2n}{n}^2=\binom{4n+1}{2n}+\binom{2n}{n}^2.
\end{aligned}$$

□