Problem 11898
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Proposed by R. Stanley (USA).

Let n and k be integers, with $n \geq k \geq 2$. Let G be a graph with n vertices whose components are cycles of length greater than k. Let $f_k(G)$ be the number of k-element independent sets of vertices of G. Show that $f_k(G)$ depends only on k and n. (A set of vertices is independent if no two of them are adjacent.)

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We will show that for $n \geq k \geq 2$,

$$f_k(G) = f_k(C_n) = \frac{n}{k}\binom{n-k-1}{k-1}$$

where C_n si the cycle of length n.
The number $kf_k(C_n)/n$, is the number of k-element independent sets of vertices of C_n where one of the vertices of the independent set is the vertex labeled 1. Then $kf_k(C_n)/n$ is equal to the number of solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n - k$$

where $x_i > 0$ is the number of vertices between two consecutive vertices of the k-element independent set. Therefore

$$f_k(C_n) = \frac{n}{k}\binom{n-k-1}{k-1}.$$

For any positive integer m, let

$$P_m(x) := 1 + \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{m}{k}\binom{m-k-1}{k-1} x^k = A^m(x) + B^m(x)$$

with

$$A(x) = \frac{1+\sqrt{1+4x}}{2} \quad \text{and} \quad B(x) = \frac{1-\sqrt{1+4x}}{2}.$$

Now $A(x)B(x) = -x$, and if $n_1 \leq n_2$ then

$$\begin{aligned} P_{n_1+n_2}(x) - P_{n_1}(x)P_{n_2}(x) &= (A^{n_1+n_2}(x) + B^{n_1+n_2}(x)) - (A^{n_1}(x) + B^{n_1}(x))(A^{n_2}(x) + B^{n_2}(x)) \\ &= -(A(x)B(x))^{n_1}(A^{n_2-n_1}(x) + B^{n_2-n_1}(x)) \\ &= -(-x)^{n_1}(A^{n_2-n_1}(x) + B^{n_2-n_1}(x)) \equiv 0 \pmod{x^{n_1}} \qquad (1) \end{aligned}$$

that is, the polynomials $P_{n_1+n_2}(x)$ and $P_{n_1}(x)P_{n_2}(x)$ coincide up to the degree $n_1 - 1$.
If the components of G are $C_{n_1}, C_{n_2}, \ldots, C_{n_r}$ with

$$n_1 + n_2 + \cdots + n_r = n \quad \text{and} \quad k < n_1 \leq n_2 \leq \cdots \leq n_r$$

then, by applying $r - 1$ times the congruence (1), we obtain

$$\begin{aligned} f_k(G) = [x^k]\prod_{j=1}^{r} P_{n_j}(x) &= [x^k]P_{n_1+n_2}(x)\prod_{j=3}^{r} P_{n_j}(x) \\ &= [x^k]P_{n_1+n_2+n_3}(x)\prod_{j=4}^{r} P_{n_j}(x) = \cdots \\ &= [x^k]P_n(x) = \frac{n}{k}\binom{n-k-1}{k-1} = f_k(C_n). \end{aligned}$$

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