Problem 11892
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Proposed by F. Perdomo and A. Plaza (Spain).

Let f be a real-valued continuously differentiable function on $[a,b]$ with positive derivative on (a,b). Prove that, for all pairs (x_1,x_2) with $a \leq x_1 < x_2 \leq b$ and $f(x_1)f(x_2) > 0$, there exists $t \in (x_1,x_2)$ such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = t - \frac{f(t)}{f'(t)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The function f is increasing in $[a,b]$, and $f(x_1)f(x_2) > 0$ implies that $f(x) \neq 0$ in $[x_1,x_2]$ (otherwise $f(x_1) \leq 0 \leq f(x_2)$). Let $F(x) := -x/f(x)$ and $G(x) := -1/f(x)$, then F and G are both continuous in $[x_1,x_2]$, and differentiable in (x_1,x_2). Moreover $G'(x) = f'(x)/f^2(x) \neq 0$ in (x_1,x_2). Then, by the Cauchy's Mean Value Theorem there exists some $t \in (x_1,x_2)$, such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{-\frac{x_2}{f(x_2)} + \frac{x_1}{f(x_1)}}{-\frac{1}{f(x_2)} + \frac{1}{f(x_1)}} = \frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)} = \frac{F'(t)}{G'(t)} = \frac{\frac{-f(t)+tf'(t)}{f^2(t)}}{\frac{f'(t)}{f^2(t)}} = t - \frac{f(t)}{f'(t)}.$$

□