Problem 11891
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Proposed by C. Chiser (Romania).

Let $\mathcal{M}$ be the set of all 3×3 matrices with complex entries. Suppose $A, B \in \mathcal{M}$ with $AB = BA$ and $\mathrm{Tr}(A^k) = \mathrm{Tr}(B^k)$ for $k \in \{1, 2, 3\}$. Suppose further that $n \geq 4$ and that $A^n - B^n$ is invertible. Prove that there exist complex numbers α, β, γ such that

$$A^{2n} + B^{2n} + A^n B^n + \alpha A^n + \beta B^n + \gamma I = 0.$$

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

We will show that the statement is true for any positive integer n.

Since A and B are commuting matrices, they are simultaneously triangularizable: there exist a invertible matrix $P \in \mathcal{M}$ such that the matrices $P^{-1}AP$ and $P^{-1}BP$ are both upper triangular. Let

$$(a_1, a_2, a_3) = \mathrm{diag}(P^{-1}AP) \quad \text{and} \quad (b_1, b_2, b_3) = \mathrm{diag}(P^{-1}BP).$$

be the complex eigenvalues of A and B.
For any matrix $M \in \mathcal{M}$, the characteristic polynomial is equal to

$$\det(xI - M) = x^3 - e_1 x^2 + e_2 x - e_3$$

with

$$e_1 = \mathrm{Tr}(M), \quad e_2 = \frac{1}{2}(\mathrm{Tr}^2(M) - \mathrm{Tr}(M^2)), \quad e_3 = \frac{1}{6}(\mathrm{Tr}^3(M) - 3\mathrm{Tr}(M)\mathrm{Tr}(M^2) + 2\mathrm{Tr}(M^3)).$$

Thus, by $\mathrm{Tr}(A^k) = \mathrm{Tr}(B^k)$ for $k \in \{1, 2, 3\}$, it follows that A and B have the same characteristic polynomial and therefore there is a permutation $\sigma \in S_3$ such that $b_k = a_{\sigma(k)}$ for $k \in \{1, 2, 3\}$.
Now, since the matrix

$$(P^{-1}AP)^n - (P^{-1}BP)^n = P^{-1}A^n P - P^{-1}B^n P = P^{-1}(A^n - B^n)P$$

is upper triangular and $A^n - B^n$ is invertible, we have that

$$\prod_{k\in\{1,2,3\}} (a_k^n - a_{\sigma(k)}^n) = \prod_{k\in\{1,2,3\}} (a_k^n - b_k^n) = \det((P^{-1}AP)^n - (P^{-1}BP)^n)$$
$$= \det(P^{-1}(A^n - B^n)P) = \det(A^n - B^n) \neq 0,$$

which imply that $\sigma(k) \neq k$ for $k \in \{1, 2, 3\}$. So σ is a cyclic permutation in S_3 and the eigenvalues a_1, a_2, a_3 are distinct complex numbers. Therefore A and B are simultaneously diagonalizable by an invertible matrix $Q \in \mathcal{M}$ and the equation to be solved is equivalent to

$$Q^{-1}(A^{2n} + B^{2n} + A^n B^n + \alpha A^n + \beta B^n + \gamma I)Q = 0,$$

which yields the linear system,

$$\begin{cases} a_1^{2n} + a_{\sigma(1)}^{2n} + a_1^n a_{\sigma(1)}^n + \alpha a_1^n + \beta a_{\sigma(1)}^n + \gamma = 0, \\ a_2^{2n} + a_{\sigma(2)}^{2n} + a_2^n a_{\sigma(2)}^n + \alpha a_2^n + \beta a_{\sigma(2)}^n + \gamma = 0, \\ a_3^{2n} + a_{\sigma(3)}^{2n} + a_3^n a_{\sigma(3)}^n + \alpha a_3^n + \beta a_{\sigma(3)}^n + \gamma = 0. \end{cases}$$

It is straightforward to verify that the above system is solved by

$$\alpha = \beta = -(a_1^n + a_2^n + a_3^n) = -t_1, \quad \gamma = a_1^n a_2^n + a_2^n a_3^n + a_3^n a_1^n = \frac{1}{2}(t_1^2 - t_2)$$

where $t_1 := \mathrm{Tr}(A^n) = \mathrm{Tr}(B^n)$ and $t_2 := \mathrm{Tr}(A^{2n}) = \mathrm{Tr}(B^{2n})$. □