Problem 11890
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Proposed by G. Apostolopoulos (Greece).

Find all $x \in (1, +\infty)$ *such that*

$$\sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{1}{x^{2k+1}} + \left(\frac{x-1}{x+1}\right)^{2k+1}\right) = \frac{1}{2}\int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It si known that

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \quad \text{for } |z|<1, \qquad \int_0^x \frac{dt}{\sqrt{1+t^2}} = \ln(x+\sqrt{1+x^2}), \quad \text{for } x \in \mathbb{R}.$$

Hence if $x > 1$ then $0 < 1/x < 1$, $0 < \frac{x-1}{x+1} < 1$ and the equation is equivalent to

$$\frac{1}{2}\ln\left(\frac{1+1/x}{1-1/x}\right) + \frac{1}{2}\ln\left(\frac{1+(x-1)/(x+1)}{1-(x-1)/(x+1)}\right) = \frac{1}{2}\ln(x+\sqrt{1+x^2})$$

or

$$\frac{x+1}{x-1}\cdot x = x + \sqrt{1+x^2}, \qquad x^4 - 2x^3 - 2x^2 - 2x + 1 = 0$$

By letting $x = t^2 + t$ with $t > 1/\varphi$ and $\varphi = (1+\sqrt{5})/2$ (the *golden ratio*), we have that

$$x^4 - 2x^3 - 2x^2 - 2x + 1 = (t^4 + 4t^3 + 5t^2 + 2t - 1)(t^4 - t^2 - 1).$$

Now $P(t) := t^4 + 4t^3 + 5t^2 + 2t - 1$ is a strictly increasing function for $t > 0$, therefore

$$P(t) > P(1/\varphi) > \frac{2}{\varphi} - 1 > 0, \qquad \text{for } t > 1/\varphi.$$

On the other hand $t^4 - t^2 - 1 = 0$ for $t > 1/\varphi$ if and only if $t = \sqrt{\varphi}$ which implies that the required equation has a unique solution in $(1, +\infty)$, namely

$$x = t^2 + t = \varphi + \sqrt{\varphi} \approx 2.890053638.$$

$\square$