Problem 11886
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Proposed by F. Holland (Ireland).

Suppose $n \geq 3$, and let $y_1, \dots, y_n$ be a list of real numbers such that $2y_{k+1} \leq y_k + y_{k+2}$ for $1 \leq k \leq n-2$. Suppose further that $\sum_{k=1}^n y_k = 0$. Prove that

$$\sum_{k=1}^n k^2 y_k \geq (n+1)\sum_{k=1}^n k y_k\,,$$

and determine when equality holds.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned}
&\sum_{k=1}^{n-2}\binom{k+1}{2}\binom{n-k}{2}(y_{k+2}-2y_{k+1}+y_k)\\
&\qquad=\sum_{k=1}^{n}\left(\binom{k+1}{2}\binom{n-k}{2}-2\binom{k}{2}\binom{n-k+1}{2}+\binom{k-1}{2}\binom{n-k+2}{2}\right)y_k\\
&\qquad=\sum_{k=1}^{n}\left(3k^2-3(n+1)k+\binom{n+2}{2}\right)y_k\\
&\qquad=3\left(\sum_{k=1}^{n}k^2y_k-(n+1)\sum_{k=1}^{n}ky_k\right)
\end{aligned}$$

where in the last step we used the fact that $\sum_{k=1}^n y_k = 0$.
Hence, the inequality follows from the hypothesis $y_{k+2} - 2y_{k+1} + y_k \geq 0$.
Moreover, since $\binom{k+1}{2}\binom{n-k}{2} > 0$, the equality holds if and only if $y_{k+2} - 2y_{k+1} + y_k = 0$ for $1 \leq k \leq n-2$, that is $y_{k+2} - y_{k+1} = y_{k+1} - y_k$ which implies that $y_k = m(k-1) + q$ with $m, q \in \mathbb{R}$.
By imposing $\sum_{k=1}^n y_k = 0$, we find that $y_k = m(k - \frac{n+1}{2})$ for $1 \leq k \leq n$. □

Remark. The following statement gives the continuous analogue of the above inequality.
Let $f \in C^2([0,1])$ be a real-valued convex function such that $\int_0^1 f(x)\,dx = 0$ then

$$\int_0^1 x^2 f(x)\,dx \geq \int_0^1 x f(x)\,dx$$

and the equality holds if and only if $f(x) = m(x-1/2)$ with $m \in \mathbb{R}$.
Infact, by using integration by parts twice, we have that

$$\begin{aligned}
\int_0^1 \frac{x^2(x-1)^2}{4} f''(x)\,dx &= -\int_0^1 \frac{x(x-1)(2x-1)}{2} f'(x)\,dx\\
&= \int_0^1 \left(3x^2-3x+\frac{1}{2}\right) f(x)\,dx = 3\left(\int_0^1 x^2 f(x)\,dx - \int_0^1 x f(x)\,dx\right).
\end{aligned}$$

Hence, the inequality follows from the convexity hypothesis $f''(x) \geq 0$. Moreover, since $x^2(x-1)^2 > 0$ in $(0,1)$, the equality holds if and only if $f''(x) = 0$ in $(0,1)$ and, together with $\int_0^1 f(x)\,dx = 0$, we obtain that $f(x) = m(x-1/2)$.