Problem 11885
(American Mathematical Monthly, Vol.123, January 2016)

Proposed by C. I. Vălean (Romania).

Prove that

$$\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^4+((m+n)(m+p))^2}=\frac{3}{2}\zeta(3)-\frac{5}{4}\zeta(4).$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let S be the left-hand side, then by setting $a=m+n$ and $b=m+p$, we have that by symmetry,

$$S=\sum_{m=1}^{\infty}\sum_{a>m}\sum_{b>m}\frac{1}{a^2(a^2+b^2)}=\sum_{m=1}^{\infty}\sum_{a>m}\sum_{b>m}\left(\frac{1}{a^2b^2}-\frac{1}{b^2(a^2+b^2)}\right)=\sum_{m=1}^{\infty}\sum_{a>m}\sum_{b>m}\frac{1}{a^2b^2}-S.$$

Hence,

$$\begin{aligned}S&=\frac{1}{2}\sum_{m=1}^{\infty}\sum_{a>m}\sum_{b>m}\frac{1}{a^2b^2}=\sum_{m=1}^{\infty}\sum_{a>b>m}\frac{1}{a^2b^2}+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{a>m}\frac{1}{a^4}\\&=\sum_{a>b\geq 1}\frac{1}{a^2b^2}\sum_{m=1}^{b-1}1+\frac{1}{2}\sum_{a\geq 1}\frac{1}{a^4}\sum_{m=1}^{a-1}1\\&=\sum_{a>b\geq 1}\frac{1}{a^2b}-\sum_{a>b\geq 1}\frac{1}{a^2b^2}+\frac{1}{2}\sum_{a\geq 1}\frac{1}{a^3}-\frac{1}{2}\sum_{a\geq 1}\frac{1}{a^4}\\&=\zeta(2,1)-\zeta(2,2)+\frac{1}{2}\zeta(3)-\frac{1}{2}\zeta(4)=\frac{3}{2}\zeta(3)-\frac{5}{4}\zeta(4),\end{aligned}$$

because the following Multiple Zeta Values are known,

$$\zeta(2,1)=\zeta(3)\qquad\text{and}\qquad\zeta(2,2)=\frac{3}{4}\zeta(4).$$

□