**Problem 11884**
(American Mathematical Monthly, Vol.123, January 2016)

Proposed by C. Lupu and T. Lupu (Romania).

*Let $f$ be a real-valued function on $[0,1]$ such that $f$ and its first two derivatives are continuous. Prove that if $f(1/2)=0$, then*

$$\int_0^1 (f''(x))^2\,dx \geq 320\left(\int_0^1 f(x)\,dx\right)^2.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Remark: this problem is a particular case of Problem 11756 with $[-1,1]$ replaced with $[0,1]$.

By the integral form of the remainder in the Taylor's Theorem we have that if $x\in[0,1]$ then

$$f(x)=f(1/2)+f'(1/2)(x-1/2)+\int_{1/2}^x f''(t)\,(x-t)\,dt.$$

Since $f(1/2)=0$, it follows that

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1\left(\int_{1/2}^x f''(t)\,(x-t)\,dt\right)dx\\
&=\int_{x=0}^{1/2}\int_{t=x}^{1/2} f''(t)\,(t-x)\,dt\,dx+\int_{x=1/2}^{1}\int_{t=1/2}^{x} f''(t)\,(x-t)\,dt\,dx\\
&=\int_{t=0}^{1/2}\int_{x=0}^{t} f''(t)\,(t-x)\,dx\,dt+\int_{t=1/2}^{1}\int_{x=t}^{1} f''(t)\,(x-t)\,dx\,dt\\
&=\int_{t=0}^{1/2} f''(t)\left[-\frac{(t-x)^2}{2}\right]_{x=0}^{t}dt+\int_{t=0}^{1} f''(t)\left[\frac{(x-t)^2}{2}\right]_{x=t}^{1}dt\\
&=\frac{1}{2}\int_{t=0}^{1/2} f''(t)t^2\,dt+\frac{1}{2}\int_{t=0}^{1} f''(t)(1-t)^2\,dt\\
&=\frac{1}{2}\int_0^1 f''(t)\,h(t)\,dt,\end{aligned}$$

where

$$h(t)=\begin{cases} t^2 & \text{if } t\in[0,1/2]\\ (1-t)^2 & \text{if } t\in[1/2,1]\end{cases}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\left(\int_0^1 f(x)\,dx\right)^2 \leq \frac{1}{4}\int_0^1 (h(t))^2\,dt\cdot\int_0^1 (f''(t))^2\,dt=\frac{1}{320}\int_0^1 (f''(t))^2\,dt$$

which is equivalent to the desired inequality. □