Problem 11883
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Proposed by H. Ohtsuka (Japan).

For $|q| > 1$, *prove that*

$$\sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{q^{2^j} + q} = \frac{1}{q-1} \prod_{j=0}^{\infty} \frac{1}{q^{1-2^j} + 1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It suffices to show that for $|x| \neq 1$ and $n \in \mathbb{N}^+$ then

$$1 + \sum_{k=1}^{n} \prod_{j=1}^{k} \frac{x^{2^j}}{1 - x^{2^j - 1}} = \frac{x^{2^{n+1}-1} + 1}{x+1} \prod_{j=1}^{n} \frac{1}{1 - x^{2^j - 1}}. \tag{1}$$

Infact, by letting $x = -1/q$ we have that $|x| < 1$ and when n goes to infinity we obtain

$$1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{q^{2^j} + q} = 1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{x^{2^j}}{1 - x^{2^j - 1}} = \frac{1}{x+1} \prod_{j=1}^{\infty} \frac{1}{1 - x^{2^j - 1}} = \frac{q}{q-1} \prod_{j=1}^{\infty} \frac{1}{1 + q^{1-2^j}}.$$

Thus, the required identity follows because

$$\sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{q^{2^j} + q} = \frac{1}{2q} \left(1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{1}{q^{2^j} + q} \right) = \frac{1}{2q} \left(\frac{q}{q-1} \prod_{j=1}^{\infty} \frac{1}{1 + q^{1-2^j}} \right) = \frac{1}{q-1} \prod_{j=0}^{\infty} \frac{1}{q^{1-2^j} + 1}.$$

Finally we prove (1) by induction. It is easy to verify it for $n = 1$.
Now assume that (1) holds for n, then it holds for $n+1$ if and only if

$$1 + \sum_{k=1}^{n+1} \prod_{j=1}^{k} \frac{x^{2^j}}{1 - x^{2^j - 1}} = \frac{x^{2^{n+1}-1} + 1}{x+1} \prod_{j=1}^{n} \frac{1}{1 - x^{2^j - 1}} + \prod_{j=1}^{n+1} \frac{x^{2^j}}{1 - x^{2^j - 1}} \stackrel{?}{=} \frac{x^{2^{n+2}-1} + 1}{x+1} \prod_{j=1}^{n+1} \frac{1}{1 - x^{2^j - 1}}$$

that is

$$\frac{x^{2^{n+1}-1} + 1}{x+1} + \frac{x^{2^{n+2}-2}}{1 - x^{2^{n+1}-1}} \stackrel{?}{=} \frac{x^{2^{n+2}-1} + 1}{(x+1)(1 - x^{2^{n+1}-1})}$$

which is true. □