Problem 11882
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Proposed by D. Callan (USA).

In a list of distinct positive integers, say that an entry a is left-full if the entries to the left of a include $1, \ldots, a-1$. *For example, the left-full entries in* 241739 *are* 1 *and* 3. *Show that the number of arrangements of* n *elements from* $\{1, 2, \ldots, 2n+1\}$ *that contain* 1 *but no other left-full entry is equal to* $(2n-1)!/n!$ *times the sum of the entries of the* $n \times n$ *Hilbert matrix* M *with* $M_{i,j} = 1/(i+j-1)$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We first count the arrangements of n elements from $\{1, 2, \ldots, 2n+1\}$ that contain 1 but no other left-full entry such that:

1) they contain the numbers $1, 2, \ldots, j$ but not the number $j+1$ with $1 \leq j \leq n$;
2) they have the number 1 at the i-th position with $j \leq i \leq n$.

In order to avoid the left-full entries, the numbers $2, \ldots, j$ should be placed before the number 1. We select the remaining $i-j$ numbers to be put before the number 1 among the remaining $2n+1-(j+1)$. This can be done in $\binom{2n-j}{i-j}$ ways. Then we permute the $i-1$ numbers before the number 1 in $(i-1)!$ ways. Finally we choose and arrange the $n-i$ numbers after the number 1 in $\binom{2n-i}{n-i}(n-i)!$ ways. Thus the number of such arrangements is

$$\binom{2n-j}{i-j}(i-1)!\binom{2n-i}{n-i}(n-i)!.$$

Therefore the total number A_n of the required arrangements is

$$\begin{aligned} A_n &= \sum_{j=1}^{n}\sum_{i=j}^{n}\binom{2n-j}{i-j}(i-1)!\binom{2n-i}{n-i}(n-i)! \\ &= \sum_{i=1}^{n}(i-1)!\binom{2n-i}{n-i}(n-i)!\sum_{j=1}^{i}\binom{2n-j}{i-j} \\ &= \sum_{i=1}^{n}(i-1)!\binom{2n-i}{n-i}(n-i)!\cdot\frac{2n}{2n-i+1}\binom{2n-1}{i-1} \\ &= \frac{(2n)!}{n!}\sum_{i=1}^{n}\frac{1}{2n-i+1} = \frac{(2n)!}{n!}(H_{2n}-H_n) \end{aligned}$$

where $H_k = \sum_{j=1}^{k} 1/j$. Since $\sum_{k=1}^{m-1} H_k = mH_m - m$, it follows that the sum of the entries of the $n \times n$ Hilbert matrix M is equal to

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j-1} &= \sum_{i=1}^{n}(H_{n+i-1}-H_{i-1}) = \sum_{i=n}^{2n-1}H_i - \sum_{i=1}^{n-1}H_i = \sum_{i=1}^{2n-1}H_i - 2\sum_{i=1}^{n-1}H_i \\ &= (2nH_{2n}-2n) - 2(nH_n - n) = 2n(H_{2n}-H_n). \end{aligned}$$

Putting all together, we obtain

$$A_n = \frac{(2n)!}{n!}(H_{2n}-H_n) = \frac{(2n-1)!}{n!}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j-1}$$

and the proof is complete. □