Problem 11880
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Proposed by D. Andrica (Romania).

Let $ABCD$ be any plane quadrilateral (not necessarily convex or even simple). Let a parallelogram be created by constructing through the ends of each diagonal of $ABCD$ lines parallel to the other diagonal. Show that each diagonal of this parallelogram passes through the intersection point of a pair of opposite sides of $ABCD$.

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The obtained parallelogram $EFGH$ can be represented in the the complex plane as

$$E = 0, \quad H = z, \quad F = w, \quad G = z + w,$$

where $z, w \in \mathbb{C}$. Then, by how the parallelogram was made, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} A &= E + \alpha(H - E) = \alpha z, & B &= E + \beta(F - E) = \beta w, \\ C &= A + (F - E) = \alpha z + w, & D &= B + (H - E) = \beta w + z. \end{aligned}$$

Then the intersection point of the lines AD and BC is given by

$$P = A + t(D - A) = B + s(C - B), \quad \text{with} \quad t = \frac{\alpha}{\alpha + \beta - 1}, \; s = \frac{\beta}{\alpha + \beta - 1}$$

which implies that

$$P = A + t(D - A) = \frac{\alpha\beta(z + w)}{\alpha + \beta - 1}.$$

Hence, the point P is on the diagonal EG because

$$\frac{P - E}{G - E} = \frac{\alpha\beta}{\alpha + \beta - 1} \in \mathbb{R}.$$

In a similar way, the intersection point of the lines AB and CD is given by

$$Q = A + t'(B - A) = C + s'(D - C), \quad \text{with} \quad t' = \frac{\alpha - 1}{\alpha - \beta}, \; s = \frac{\alpha}{\alpha - \beta}$$

which implies that

$$Q = A + t'(B - A) = \frac{\alpha(1 - \beta)z - \beta(1 - \alpha)w}{\alpha - \beta}.$$

Hence, the point Q is on the diagonal FH because

$$\frac{Q - F}{H - F} = \frac{\alpha(1 - \beta)}{\alpha - \beta} \in \mathbb{R}.$$