Problem 11879
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Proposed by S. Siboni (Italy).

For positive a, b, and c, prove that there exist positive α, β, and γ with $\alpha+\beta+\gamma=\pi$ such that

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

if and only if $|b-c|<a<b+c$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

If $|b-c|<a<b+c$ then

$$a<b+c,\quad b=b-c+c\leq |b-c|+c<a+c,\quad c=c-b+b\leq |b-c|+b<a+b$$

which imply that there exists a non-degenerate triangle with a, b, c as its side lengths.
Let α, β, γ the radian measures of the corresponding opposite angles, then they are positive numbers such that $\alpha+\beta+\gamma=\pi$ and, by the law of sines,

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}.$$

Now assume that there are positive α, β, and γ with $\alpha+\beta+\gamma=\pi$ such that

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}.$$

Then $b=a\sin\beta/\sin\alpha$, $c=a\sin\gamma/\sin\alpha$ and it suffices to show that

$$|\sin\beta-\sin\gamma|<\sin\alpha<\sin\beta+\sin\gamma$$

that is

$$\sin\alpha<\sin\beta+\sin\gamma,\quad \sin\beta<\sin\alpha+\sin\gamma,\quad \sin\gamma<\sin\alpha+\sin\beta.$$

We show the first inequality:

$$\sin\alpha=\sin(\pi-\beta-\gamma)=\sin(\beta+\gamma)=\sin\beta\cos\gamma+\sin\gamma\cos\beta<\sin\beta+\sin\gamma$$

because $\sin\beta>0$, $\sin\gamma>0$, $\cos\beta<1$, and $\cos\gamma<1$. The other two can be verified in a similar way. $\square$