Problem 11875
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Proposed by D. M. Bătineţu-Girugiu and N. Stanciu (Romania).

Let $f_n = (1+1/n)^n((2n-1)!!\,L_n)^{1/n}$*. Find* $\lim_{n\to\infty}(f_{n+1}-f_n)$ *where* L_n *denotes the* n*th Lucas number (given by* $L_0 = 2$*,* $L_1 = 1$*, and by* $L_n = L_{n-1}+L_{n-2}$ *for* $n \geq 2$*.*

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

It is known that,

$$\left(1+\frac{1}{n}\right)^n = e - \frac{e}{2n} + o(1/n),$$
$$((2n-1)!!)^{1/n} = \left(\frac{(2n)!}{2^n n!}\right)^{1/n} = \frac{2n}{e} + \frac{\ln(2)}{e} + o(1),$$
$$L_n^{1/n} = \left(\tau^n + \frac{1}{\tau^n}\right)^{1/n} = \tau + o(1/n).$$

where $\tau = (1+\sqrt{5})/2$. Hence $f_n = 2\tau n - \tau(\ln 2 - 1) + o(1)$ and it follows that

$$\lim_{n\to\infty}(f_{n+1}-f_n) = 2\tau = 1+\sqrt{5}.$$

□