Problem 11874
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Evaluate the limits below,

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)},\qquad \lim_{n\to\infty}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{\zeta(n-k)}{\Gamma(k)}.$$

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

We first note that

$$\sum_{k=2}^{\infty}(\zeta(k)-1)=\sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n^k}=\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{n^k}=\sum_{n=2}^{\infty}\frac{1/n^2}{1-1/n}=\sum_{n=2}^{\infty}\frac{1}{n(n-1)}=\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)=1.$$

Moreover for all $z\in\mathbb{C}$,

$$\sum_{k=0}^{\infty}\frac{z^{k-1}}{\Gamma(k)}=\sum_{k=1}^{\infty}\frac{z^k}{k!}=e^z.$$

The Cauchy product of these two absolutely convergent series is

$$\sum_{n=3}^{\infty}\left(\sum_{k=2}^{n-1}\frac{(\zeta(k)-1)z^{n-k+1}}{\Gamma(n-k)}\right)=e^z$$

where the convergence is given by Mertens' theorem.
Since the general term of the series goes to zero, we have that

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{(\zeta(k)-1)z^{n-k+1}}{\Gamma(n-k)}=0.$$

Therefore

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)z^{n-k+1}}{\Gamma(n-k)}=\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{z^{n-k+1}}{\Gamma(n-k)}+\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{(\zeta(k)-1)z^{n-k+1}}{\Gamma(n-k)}=\lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{z^{k-1}}{\Gamma(k)}=e^z.$$

Finally we apply this result for $z=1$ and $z=-1$ to the given limits,

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)}\overset{z=1}{=}e\quad\text{and}\quad\lim_{n\to\infty}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{\zeta(n-k)}{\Gamma(k)}=\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)(-1)^{n-k-1}}{\Gamma(n-k)}\overset{z=-1}{=}e^{-1}.$$

□