Problem 11873
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Proposed by E. J. Ionascu (USA).

Show that for $n \in \mathbb{N}$ with $n \geq 2$,

$$\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\cot\frac{(2k-1)\pi}{2n}=\sum_{k=1}^{n-1}\csc\frac{k\pi}{n}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

First solution. We will show the identity by computing the integral

$$I_n := \int_0^1 \frac{1-x^{n-1}}{(1-x)(1+x^n)}\,dx$$

in two different ways.
By using the partial fraction decomposition, for $x \geq 0$,

$$\begin{aligned}\frac{1-x^{n-1}}{(1-x)(1+x^n)} &= \sum_{k=1}^{n}\frac{\frac{1-w_k^{n-1}}{(1-w_k)nw_k^{n-1}}}{x-w_k} = -\frac{i}{n}\sum_{k=1}^{n}\frac{\cot(\theta_k/2)}{x-w_k}\\ &= \frac{1}{n}\sum_{k=1}^{n}\cot(\theta_k/2)\cdot\operatorname{Im}\left(\frac{1}{x-w_k}\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\cot(\theta_k/2)\sin\theta_k}{(x-\cos\theta_k)^2+\sin^2\theta_k}\end{aligned}$$

where $w_k = \exp(i\theta_k)$ with $\theta_k = (2k-1)\pi/n$ for $k = 1, \ldots, n$ are the n-th roots of -1. Hence

$$\begin{aligned}I_n &= \frac{1}{n}\sum_{k=1}^{n}\cot(\theta_k/2)\int_0^1\frac{\sin\theta_k}{(x-\cos\theta_k)^2+\sin^2\theta_k}\,dx\\ &= \frac{1}{n}\sum_{k=1}^{n}\cot(\theta_k/2)\left[\arctan\left(\frac{x-\cos\theta_k}{\sin\theta_k}\right)\right]_0^1\\ &= \frac{\pi}{2n}\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\cot\frac{(2k-1)\pi}{2n}.\end{aligned}$$

On the other hand, we have that,

$$\begin{aligned}I_n &= \int_0^1\frac{\sum_{k=1}^{n-1}x^{k-1}}{1+x^n}\,dx = \frac{1}{2}\sum_{k=1}^{n-1}\left(\int_0^1\frac{x^{k-1}}{1+x^n}\,dx+\int_0^1\frac{x^{n-k-1}}{1+x^n}\,dx\right)\\ &= \frac{1}{2}\sum_{k=1}^{n-1}\left(\int_0^1\frac{x^{k-1}}{1+x^n}\,dx+\int_{+\infty}^1\frac{(1/x)^{n-k-1}}{1+(1/x)^n}\,d(1/x)\right)\\ &= \frac{1}{2}\sum_{k=1}^{n-1}\left(\int_0^1\frac{x^{k-1}}{1+x^n}\,dx+\int_1^{+\infty}\frac{x^{k-1}}{1+x^n}\right)\\ &= \frac{1}{2}\sum_{k=1}^{n-1}\int_0^{+\infty}\frac{x^{k-1}}{1+x^n}\,dx = \frac{\pi}{2n}\sum_{k=1}^{n-1}\csc\frac{k\pi}{n}.\end{aligned}$$

The proof is complete as soon as we compare the two formulas for I_n. $\square$

Second solution. It is known that for $z = e^{i\theta}$ with $\theta \in \mathbb{R}$,

$$\cot\theta = i\,\frac{z+z^{-1}}{z-z^{-1}} = i + \frac{2i}{z^2-1} \quad \text{and} \quad \csc\theta = \frac{2i}{z-z^{-1}} = \frac{2iz}{z^2-1}$$

Hence, by letting $z = e^{i\pi/n}$, the identity is equivalent to

$$\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\left(i+\frac{2i}{z^{2k-1}-1}\right) = \sum_{k=1}^{n-1}\frac{2iz^k}{z^{2k}-1}$$

that is

$$\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\cdot\frac{1}{z^{2k-1}-1} = \sum_{k=1}^{n-1}\frac{z^k}{z^{2k}-1}.$$

The above identity holds because

$$\begin{aligned}
\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\cdot\frac{1}{z^{2k-1}-1} &= \sum_{k=1}^{2n-1}\left(1-\frac{k}{n}\right)\cdot\frac{1}{z^k-1} - \sum_{k=1}^{n-1}\left(1-\frac{2k}{n}\right)\cdot\frac{1}{z^{2k}-1}\\
&= \sum_{k=1}^{2n-1}\frac{1}{z^k-1} - \frac{1}{n}\sum_{k=1}^{2n-1}\frac{k}{z^k-1} - \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{z^k-1} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{z^k+1}\\
&\quad + \frac{1}{n}\sum_{k=1}^{n-1}\frac{k}{z^k-1} - \frac{1}{n}\sum_{k=1}^{n-1}\frac{k}{z^k+1}\\
&= \sum_{k=n}^{2n-1}\frac{1}{z^k-1} - \frac{1}{n}\sum_{k=n}^{2n-1}\frac{k}{z^k-1} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{z^k-1} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{z^k+1}\\
&\quad - \frac{1}{n}\sum_{k=n}^{2n-1}\frac{k-n}{z^{k-n}+1}\\
&\overset{z^n=-1}{=} \sum_{k=n}^{2n-1}\frac{1}{z^k-1} - \frac{1}{n}\sum_{k=n}^{2n-1}\frac{k}{z^k-1} + \sum_{k=1}^{n-1}\frac{z^k}{z^{2k}-1} + \frac{1}{n}\sum_{k=n}^{2n-1}\frac{k-n}{z^k-1}\\
&= \sum_{k=1}^{n-1}\frac{z^k}{z^{2k}-1}.
\end{aligned}$$

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