Problem 11872
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Let f be a continuous function from $[0,1]$ into $\mathbb{R}$ such that $\int_0^1 f(x)dx = 0$. Prove that for all positive integers n there exists $c \in (0,1)$ such that

$$n \int_0^c x^n f(x)dx = c^{n+1} f(c).$$

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

We will show a more general result.
Let $f \in C[0,1]$ such that $\int_0^1 f(x)dx = 0$ and let $u \in C^1[0,1]$ such that $u(0) = 0$, and $u'(x) > 0$ for $x \in (0,1)$. Then there exists $c \in (0,1)$ such that

$$\frac{u'(c)}{u(c)^2} \int_0^c u(x)f(x)dx = f(c).$$

The original problem can be obtained by letting $u(x) = x^n$.
Let us consider the function

$$T(t) = \frac{1}{u(t)} \int_0^t u(x)f(x)dx$$

which is well defined and differentiable in $(0,1)$.
Claim 1. $\lim_{t\to 0^+} T(t) = 0$.
By the Mean Value Theorem, for $t \in (0,1)$ there exists $s_t \in (0,t)$ such that

$$0 \le |T(t)| = \frac{1}{u(t)} \left| \int_0^t u(x)f(x)dx \right| = \frac{u(s_t)|f(s_t)|t}{u(t)} \le |f(s_t)|t$$

where in the last inequality we used the fact that u is increasing in $[0,1]$. Hence, as $t \to 0^+$, it follows that $|f(s_t)| \to |f(0)|$ and $T(t) \to 0$.
Claim 2. There is a $x_0 \in (0,1)$ such that $T(x_0) = 0$.
If the claim is false, without loss of generality, we can assume that $T(t) > 0$ for all $t \in (0,1)$.
Let $F(t) = \int_0^t f(x)dx$ then for $t \in (0,1)$,

$$T(t) = \frac{1}{u(t)} \int_0^t u(x)f(x)dx = \frac{1}{u(t)} [u(x)F(x)]_0^t - \frac{1}{u(t)} \int_0^t u'(x)F(x)dx = F(t) - G(t)$$

where $G(t) = \frac{1}{u(t)} \int_0^t u'(x)F(x)dx$. G is differentiable in $(0,1)$ and

$$G'(t) = \frac{u(t)u'(t)F(t) - u'(t)\int_0^t u'(x)F(x)dx}{u(t)^2} = \frac{u'(t)(F(t) - G(t))}{u(t)} = \frac{u'(t)T(t)}{u(t)} > 0.$$

By Claim 1, $\lim_{t\to 0^+} G(t) = \lim_{t\to 0^+}(F(t) - T(t)) = 0$ and it follows that $\lim_{t\to 1^-} G(t) > 0$.
On the other hand, the hypothesis $F(1) = 0$ and $T(t) > 0$ imply that $\lim_{t\to 1^-} G(t) = \lim_{t\to 1^-}(F(t) - T(t)) \le 0$. So, we get a contradiction.
Claim 3. The above statement holds.
By Claim 1, T can be extended in a continuous way at 0 by letting $T(0) = 0$. Moreover, by Claim 2, $T(x_0) = 0$. Therefore, by Rolle's Theorem there is $c \in (0,x_0) \subset (0,1)$ such that

$$0 = T'(c) = -\frac{u'(c)}{u(c)^2} \int_0^c u(x)f(x)dx + f(c).$$

□