Problem 11871
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Let ABC be a triangle in the Cartesian plane with vertices in $\mathbb{Z}^2$. Show that, if P is an interior lattice point of ABC, then at least one of the angles PAB, PBC, and PCA has a radian measure that is not a rational multiple of π.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Preliminary remarks:
- if θ is the radian measure of an angle of a triangle with vertices in $\mathbb{Z}^2$ then $\theta = \pi/2$ or $\tan(\theta) \in \mathbb{Q}$;
- the only rational values of $\tan(q\pi)$ for $q \in \mathbb{Q}$ are 0 and ± 1.

Assume that $PAB = q_1\pi$, $PBC = q_2\pi$, and $PCA = q_3\pi$ where $q_1, q_2, q_3 \in \mathbb{Q}$.
Moreeover $q_1, q_2, q_3 > 0$ and $q_1 + q_2 + q_3 < 1$ because P is an interior point.
Since the triangles PAB, PBC, and PCA have vertices in $\mathbb{Z}^2$, by the above remarks it follows that $\tan(q_1\pi) = \tan(q_2\pi) = \tan(q_3\pi) = 1$, that is $q_1 = q_2 = q_3 = 1/4$. Hence

$$CAP + ABP + BCP = \pi - (PAB + PBC + PCA) = \pi/4$$

which implies

$$\sin(CAP)\sin(ABP)\sin(BCP) < \sin(\pi/4)^3 = \frac{1}{2\sqrt{2}}.$$

On the other hand, by Ceva's theorem,

$$\sin(CAP)\sin(ABP)\sin(BCP) = \sin(PAB)\sin(PBC)\sin(PCA) = \frac{1}{2\sqrt{2}}$$

and we have a contradiction. □