Problem 11869
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Prove that $|y\log y - x\log x| \leq |y-x|^{1-1/e}$ *for* $0 < x < y \leq 1$.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Let $t = y - x \in (0,1)$ and $x \in (0, 1-t]$, we have to show that $|\log(f_t(x))| \leq t^{1-1/e}$ where $f_t(x) = (x+t)^{(x+t)}/x^x$. Since

$$\frac{d(f_t(x))}{dx} = f_t(x)\ln(1+\frac{t}{x}) > 0,$$

it follows that the function $\log(f_t(x))$ is increasing with respect to x. Hence it suffices to show that

$$\text{i)}\ -t\ln(t) = \lim_{x\to 0^+} |\log(f_t(x))| \leq t^{1-1/e} \quad \text{and} \quad \text{ii)}\ -(1-t)\ln(1-t) = |\log(f_t(1-t))| \leq t^{1-1/e}.$$

i) The first inequality is equivalent to

$$g(t) := t^{-1/e} + \ln(t) \geq 0 = g(e^{-e})$$

which holds because $g'(t) = ((-1/e)t^{-1/e}+1)/t$, so g is decreasing in $(0, e^{-e}]$ and it is increasing in $[e^{-e}, 1)$.
ii) The second inequality is equivalent to

$$h(t) := t^{1-1/e} + (1-t)\ln(1-t) \geq 0 = h(0)$$

which holds because

$$h'(t) = (1-1/e)t^{-1/e} - \ln(1-t) - 1 \geq (1-1/e)t^{-1/e} + t - 1 \geq (1/2)t^{-1/3} + t - 1 \geq (2/3)6^{1/4} - 1 > 0,$$

so h is increasing in $(0,1)$. □