Problem 11867
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Proposed by George Apostolopoulos (Greece).

For real numbers a, b, c, let

$$f(a,b,c) = \left(\frac{a^2}{a^2-ab+b^2}\right)^{1/4}.$$

Prove that $f(a,b,c)+f(b,c,a)+f(c,a,b) \leq 3$.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, Italy.

Since $f(a,b,c) \leq f(|a|,|b|,|c|)$, it suffices to consider the case where $a,b,c>0$.
Let $x=b/a$, $y=c/b$, $z=a/c$ and let $g(t)=1/(1-t+t^2)$ then

$$f(a,b,c)+f(b,c,a)+f(c,a,b) = g(x)^{1/4}+g(y)^{1/4}+g(z)^{1/4} \geq 3\left(\frac{g(x)+g(y)+g(z)}{3}\right)^{1/4}$$

because $x \to x^{1/4}$ is concave in $[0,+\infty)$.
Hence it suffices to show that for $x,y,z>0$, if $xyz=1$ then $g(x)+g(y)+g(z) \leq 3$. We will show that the following stronger inequality holds

$$g(x)+g(y)+g(z)+(1-xg(x))(1-yg(y))(1-zg(z)) \leq 3$$

(notice that $1-tg(t)=(1-t)^2/(1-t+t^2) \geq 0$). By expanding and by taking account of the constraint $xyz=1$, the above inequality is equivalent to

$$x^2y^2+y^2z^2+z^2x^2-3(xy+yz+zx)+6 \geq 0,$$

or, by letting $X=1/x$, $Y=1/y$, $Z=1/z$, to

$$h(X,Y,Z) := X^2+Y^2+Z^2-3(X+Y+Z)+6 \geq 0.$$

In order to prove it, we use the mixing variables method.
Without loss of generality we can assume that $Z \leq 1$, then $XY=1/Z \geq 1$ and

$$h(X,Y,Z) = h(\sqrt{XY},\sqrt{XY},Z)+(X-Y)^2-3(\sqrt{X}-\sqrt{Y})^2 \geq h(\sqrt{XY},\sqrt{XY},Z)$$

because $(\sqrt{X}+\sqrt{Y})^2 \geq 4\sqrt{XY} \geq 4$ and

$$(X-Y)^2-3(\sqrt{X}-\sqrt{Y})^2 = (\sqrt{X}-\sqrt{Y})^2((\sqrt{X}+\sqrt{Y})^2-3) \geq 0.$$

Finally, $Z \leq 1$ implies

$$h(\sqrt{XY},\sqrt{XY},Z) = h(1/\sqrt{Z},1/\sqrt{Z},Z) = (Z^2+2Z\sqrt{Z}+2(1-\sqrt{Z}))(\sqrt{Z}-1)^2/Z \geq 0.$$

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