Problem 11865
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Proposed by Gary H. Chung (USA).

Let $\{a_n\}_{n\geq 1}$ be a monotone decreasing sequence nonnegative real numbers.
Prove that $\sum_{n=1}^{\infty} a_n/n < \infty$ if and only if $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty}(a_n - a_{n+1})\log(n) < \infty$.

Solution proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France, and Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy.

Let

$$S_N = \sum_{n=1}^{N} \frac{a_n}{n} \qquad \text{and} \qquad T_N = \sum_{n=1}^{N} (a_n - a_{n+1})\log(n).$$

If $\{a_n\}_{n\geq 1}$ is a monotone decreasing sequence nonnegative real numbers then $\{S_N\}_{N\geq 1}$ and $\{T_N\}_{N\geq 1}$ are increasing. Let S and T their limits. Notice that for $n \geq 2$,

$$\log(n) - \log(n-1) = \int_{n-1}^{n} \frac{dx}{x} \in \left(\frac{1}{n}, \frac{1}{n-1}\right).$$

i) If $S < \infty$ then for $N \geq 2$,

$$a_N \log(N) \leq a_N \sum_{n=2}^{N} (\log(n) - \log(n-1)) \leq a_N \sum_{n=2}^{N} \frac{1}{n-1} \leq \sum_{n=2}^{N} \frac{a_{n-1}}{n-1} = S_{N-1} \leq S$$

which implies that $a_n = O(1/\log(n))$ and therefore $\lim_{n\to\infty} a_n = 0$. Moreover

$$\begin{aligned} T_N &= \sum_{n=1}^{N} (a_n - a_{n+1})\log(n) = \sum_{n=2}^{N} a_n \log(n) - \sum_{n=2}^{N+1} a_n \log(n-1) \\ &= \sum_{n=2}^{N} a_n(\log(n) - \log(n-1)) - a_n \log(n-1) \\ &\leq \sum_{n=2}^{N} \frac{a_n}{n-1} \leq \sum_{n=2}^{N} \frac{a_{n-1}}{n-1} = S_{N-1} \leq S \end{aligned}$$

which implies that $T < \infty$.

ii) If $T < \infty$ and $\lim_{n\to\infty} a_n = 0$ then for $k \geq N \geq 2$,

$$\log(N)(a_N - a_{k+1}) = \log(N) \sum_{n=N}^{k} (a_n - a_{n+1}) \leq \sum_{n=N}^{k} \log(n)(a_n - a_{n+1}) = T_k - T_{N-1} \leq T$$

and as $k \to \infty$, we obtain that $a_N \log(N) \leq T$. Hence

$$\begin{aligned} S_N - a_1 &= \sum_{n=2}^{N} \frac{a_n}{n} \leq \sum_{n=2}^{N} a_n(\log(n) - \log(n-1)) = \sum_{n=2}^{N} a_n \log(n) - \sum_{n=2}^{N-1} a_{n+1} \log(n) \\ &= \sum_{n=2}^{N-1} (a_n - a_{n+1})\log(n) + a_N \log(N) = T_{N-1} + a_N \log(N) \leq 2T, \end{aligned}$$

which implies that $S < \infty$.