Problem 11864
(American Mathematical Monthly, Vol.122, October 2015)

Proposed by Bakir Farhi (Algeria).

Let p be a prime number, and let $\{u_n\}_{n\geq 0}$ be the sequence given by $u_n = n$ for $0 \leq n \leq p-1$ and by $u_n = pu_{n+1-p} + u_{n-p}$ for $n \geq p$. Prove that for each positive integer n, the greatest power of p dividing u_n is the same as the greatest power of p dividing n.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $\nu_p(n)$ be the greatest power of p dividing a positive integer n.
We will show a more general statement: if $s = \nu_p(n)$ then $u_n \equiv n \pmod{p^{s+1}}$.

From the generating function of the sequence $\{u_n\}_{n\geq 0}$ we obtain

$$\begin{aligned} u_n &= [x^n]\frac{\sum_{k=1}^{p-1} kx^k}{1-(px^{p-1}+x^p)} = \sum_{k=1}^{p-1} k[x^{n-k}]\sum_{j\geq 0}(px^{p-1}+x^p)^j \\ &= \sum_{k=1}^{p-1} k\sum_{j\geq 0}[x^{n-k-pj}](1+p/x)^j = \sum_{k=1}^{p-1} k\sum_{j\geq 0}\binom{j}{pj+k-n}p^{pj+k-n}. \end{aligned}$$

If $n = qp + r$ with $1 \leq r \leq p-1$ then $pj + k - n = 0$ iff $j = q$ and $k = r$, hence $u_n \equiv r \equiv n \pmod{p}$.
If $n = ap^s$ with $s \geq 1$ and $\gcd(a,p) = 1$ then $pj + k - n \geq 0$ iff $j \geq \lceil (n-k)/p \rceil = ap^{s-1}$ and, by letting $i = j - ap^{s-1}$, we have

$$u_n = \sum_{k=1}^{p-1} k\sum_{i\geq 0}\binom{ap^{s-1}+i}{pi+k}p^{pi+k}.$$

Then, for $i \geq 1$ and $1 \leq k \leq p-1$,

$$\nu_p((pi+k)!) = \nu_p((pi)!) = i + \nu_p(i!) \leq i + \nu_2(i!) = i + \sum_{j=1}^{\lfloor \ln_2(i)\rfloor}\lfloor i/2^j\rfloor < i + \sum_{j=1}^{\infty} i/2^j = 2i,$$

which implies that $\nu_p((pi+k)!) \leq 2i - 1$. Hence

$$\nu_p\left(k\binom{ap^{s-1}+i}{pi+k}p^{pi+k}\right) \geq \nu_p(ap^{s-1}) - \nu_p((pi+k)!) + pi + k \geq (s-1) - (2i-1) + pi + 1 = s+1$$

where in the last step we used the fact that $p \geq 2$. Finally

$$u_n \equiv \sum_{k=1}^{p-1} k\binom{ap^{s-1}}{k}p^k = ap^{s-1}\sum_{k=1}^{p-1}\binom{ap^{s-1}-1}{k-1}p^k \equiv ap^s = n \pmod{p^{s+1}}.$$

□