Problem 11863
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Proposed by Jeffrey C. Lagarias and Jeffrey Sun (USA).

Consider integers a, b, c with $1 \leq a < b < c$ that satisfy the following system of congruences:

$$\begin{aligned}(a+1)(b+1) &\equiv 1 \pmod{c}\\ (a+1)(c+1) &\equiv 1 \pmod{b}\\ (b+1)(c+1) &\equiv 1 \pmod{a}.\end{aligned}$$

(a) Show that there are infinitely many solutions (a,b,c) to this system.
(b) Show that under the additional condition that $\gcd(a,b)=1$, there are only finitely many solutions (a,b,c) to the system, and find them all.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

(a) For any $a \geq 1$, take $b=(a+1)^2-1$, $c=(a+1)^3-1$ then

$$\begin{aligned}(a+1)(b+1) &= (a+1)^3 = c+1 \equiv 1 \pmod{c}\\ (a+1)(c+1) &\equiv (a+1)^4 = (b+1)^2 \equiv 1 \pmod{b}\\ (b+1)(c+1) &= (a+1)^5 \equiv 1 \pmod{a}.\end{aligned}$$

Hence we have an infinite set of solutions

$$\{(1,3,7),(2,8,26),(3,15,63),(4,24,124),(5,35,215),(6,48,342),(7,63,511),(8,80,728),(9,99,999),\dots\}.$$

Note that for these solutions, $\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=a$.
(b) Assume that $\gcd(a,b)=1$ and let $d=\gcd(a,c)$ then $d \mid a$ and

$$(a+1)(b+1) \equiv 1 \pmod{c} \Rightarrow ab+a+b \equiv 0 \pmod{c} \Rightarrow b \equiv ab+a+b \equiv 0 \pmod{d} \Rightarrow d \mid b.$$

Hence $d \mid \gcd(a,b)=1$ and $d=1$. In a similar way, we get that $\gcd(b,c)=1$.
Now the three congruences imply

$$\begin{aligned}(a+1)(b+1)(c+1) &\equiv 1 \pmod{c}\\ (a+1)(b+1)(c+1) &\equiv 1 \pmod{b}\\ (a+1)(b+1)(c+1) &\equiv 1 \pmod{a}\end{aligned}$$

and since $\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=1$, it follows that

$$m := \frac{(a+1)(b+1)(c+1)-1}{abc} = 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac} > 1$$

is an integer. Therefore $m \geq 2$. Moreover, $a \geq 1$, $b \geq 2$ and $c \geq 3$ imply that

$$4abc = (2a)(3b/2)(4c/3) \geq (a+1)(b+1)(c+1) = mabc+1 \Rightarrow 4-\frac{1}{abc} \geq m \Rightarrow m \leq 3.$$

If $m=2$ then we have $(a+1)(b+1)(c+1)=2abc+1$, which implies that

$$a+1 \equiv b+1 \equiv c+1 \equiv 1 \pmod{2} \Rightarrow a \equiv b \equiv c \equiv 0 \pmod{2}$$

in contradiction with the fact that $\gcd(a,b)=\gcd(a,c)=\gcd(b,c)=1$.
If $m=3$ and $a \geq 2$ then $b \geq 3$, $c \geq 4$ and

$$3 \leq \frac{(2+1)(3+1)(4+1)-1}{2\cdot 3\cdot 4} = \frac{59}{24} < 3.$$

Therefore $m=3$ and $a=1$:

$$3 = \frac{2(b+1)(c+1)-1}{bc} \Rightarrow (b-2)(c-2)=5 \Rightarrow b=3,\ c=7.$$

The conclusion is that if $\gcd(a,b)=1$ then the system has a unique solution: $(a,b,c)=(1,3,7)$. $\square$