**Problem 11862**
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Proposed by David A. Cox and Uyen Thieu (USA).

*For positive integers $n$ and $k$, evaluate*

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{kn-in}{k+1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$[t^{k+1}]((1+t)^n-1)^k=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[t^{k+1}](1+t)^{n(k-i)}=\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{kn-in}{k+1}.$$

On the other hand

$$\begin{aligned}[t^{k+1}]((1+t)^n-1)^k&=[t^{k+1}]\left(t\sum_{j=1}^{n}\binom{n}{j}t^{j-1}\right)^k=[t]\left(\sum_{j=1}^{n}\binom{n}{j}t^{j-1}\right)^k\\&=[t]\left(n+\binom{n}{2}t+o(t)\right)^k=n^k[t]\left(1+\frac{n-1}{2}t+o(t)\right)^k\\&=\frac{kn^k(n-1)}{2}.\end{aligned}$$

Therefore

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{kn-in}{k+1}=\frac{kn^k(n-1)}{2}.$$

□