Problem 11861
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Proposed by Phu Cuong Le Van (Vietnam).

Let n be a natural number and let f be a continuous function from $[0,1]$ to $\mathbb{R}$ such that $\int_0^1 f(x)^{2n+1}dx = 0$. Prove that

$$\frac{(2n+1)^{2n+1}}{(2n)^{2n}}\left(\int_0^1 f(x)dx\right)^{4n} \leq \int_0^1 (f(x))^{4n}dx.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $t \in \mathbb{R}$, by Cauchy-Schwarz inequality,

$$\left(\int_0^1 (t + f(x)^{2n})f(x)dx\right)^2 \leq \int_0^1 (t + f(x)^{2n})^2 dx \int_0^1 f(x)^2 dx,$$

Let $a_k = \int_0^1 f(x)^k dx$ and assume that f is not identically zero (otherwise the inequality is trivial). Since $a_{2n+1} = 0$ and $a_2 > 0$, it follows that

$$a_1^2 t^2 = (a_1 t + a_{2n+1})^2 \leq (t^2 + 2a_{2n}t + a_{4n})a_2,$$

that is

$$\left(\frac{a_1^2}{a_2} - 1\right)t^2 - 2a_{2n}t \leq a_{4n}.$$

By Cauchy-Schwarz inequality $a_1^2 \leq a_2$. More precisely, we have that $a_1^2 < a_2$ because equality holds iff f is constant which is in contradiction with the facts: $a_{2n+1} = 0$ and $a_2 > 0$.
Now the LHS is a concave quadratic function with respect to t and it attains its maximum value at $t = a_{2n}/\left(\frac{a_1^2}{a_2} - 1\right)$. Hence, we get

$$\frac{a_2^{2n+1}}{(a_2 - a_1^2)} \overset{\text{CS}}{\leq} \frac{a_{2n}^2}{\left(1 - \frac{a_1^2}{a_2}\right)} \leq a_{4n}.$$

The LHS is a convex function with respect to a_2 for $a_2 > a_1^2$ and it attains its minimum value at $a_2 = (2n+1)a_1^2/(2n)$. Therefore

$$\frac{(2n+1)^{2n+1}}{(2n)^{2n}}\, a_1^{4n} = \frac{((2n+1)a_1^2/(2n))^{2n+1}}{((2n+1)a_1^2/(2n) - a_1^2)} \leq a_{4n}$$

and the proof is complete. $\square$