Problem 11857
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Let ABC be a triangle with corresponding sides of lengths a, b, and c, inradius r, and corresponding exradii r_a, r_b, and r_c. Let $A'B'C'$ be another triangle with sides of lengths $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. Show that $A'B'C'$ has area given by

$$\frac{1}{2}\sqrt{r(r_a+r_b+r_c)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $x=\sqrt{a}$, $y=\sqrt{b}$, and $z=\sqrt{c}$ then $a+b\geq c$ implies that

$$x+y=\sqrt{a}+\sqrt{b}\geq\sqrt{a+b}=\sqrt{c}=z.$$

Hence the triangle $A'B'C'$ is well defined. We have that

$$r=\frac{\Delta}{s},\quad r_a=\frac{\Delta}{s-a},\quad r_b=\frac{\Delta}{s-b},\quad r_c=\frac{\Delta}{s-c},$$

where Δ and s are the area and the perimeter of ABC respectively. Then

$$\begin{aligned}
r(r_a+r_b+r_c)&=\frac{\Delta^2}{s}\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\\
&=(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)\\
&=\frac{1}{2}\left((s-c)(2s-a-b)+(s-b)(2s-a-c)+(s-a)(2s-b-c)\right)\\
&=\frac{1}{2}\left((s-c)c+(s-b)b+(s-a)a\right)\\
&=\frac{1}{4}\left((a+b+c)^2-2a^2-2b^2-2c^2\right)\\
&=\frac{1}{4}\left((x^2+y^2+z^2)^2-2x^4-2y^4-2z^4\right)\\
&=\frac{1}{4}(x+y+z)(y+z-x)(z+x-y)(x+y-z)=(2\Delta')^2
\end{aligned}$$

where Δ' is the area of $A'B'C'$. □