Problem 11855
(American Mathematical Monthly, Vol.122, August-September 2015)

Proposed by C. Lupu (USA).

For a continuous and nonnegative function f on $[0,1]$, let $\mu_n = \int_0^1 x^n f(x)dx$. Show that $\mu_{n+1}\mu_0 \geq \mu_n\mu_1$ for $n \in \mathbb{N}$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We show a more general result (which is particular case of Chebyshev's inequality): let G and H two continuous functions on $[0,1]$, both increasing or both decreasing then

$$\int_0^1 G(x)H(x)f(x)dx \int_0^1 f(x)dx \geq \int_0^1 G(x)f(x)dx \int_0^1 H(x)f(x)dx.$$

By taking $G(x) = x^n$ and $H(x) = x$ we obtain the result.

If f is identically zero then the inequality is trivial, otherwise f is non-negative and positive at some point in $[0,1]$ which implies that $\mu_0 = \int_0^1 f(x)dx > 0$.
We consider the case where G and H are both increasing (the decreasing case is similar). Then

$$\mu_0 H(0) \leq \int_0^1 H(x)f(x)dx \leq \mu_0 H(1)$$

and by continuity there exists $x_0 \in [0,1]$ such that

$$H(x_0) = \frac{1}{\mu_0}\int_0^1 H(x)f(x)dx.$$

Therefore $(G(x) - G(x_0))(H(x) - H(x_0)) \geq 0$ for all $x \in [0,1]$ because G and H are both increasing, which implies that

$$\begin{aligned}
0 &\leq \int_0^1 (G(x) - G(x_0))(H(x) - H(x_0))f(x)dx \\
&= \int_0^1 G(x)H(x)f(x)dx - G(x_0)\int_0^1 H(x)f(x)dx \\
&\qquad - H(x_0)\int_0^1 G(x)f(x)dx + G(x_0)H(x_0)\mu_0 \\
&= \int_0^1 G(x)H(x)f(x)dx - H(x_0)\int_0^1 G(x)f(x)dx \\
&= \int_0^1 G(x)H(x)f(x)dx - \frac{1}{\mu_0}\int_0^1 G(x)f(x)dx \int_0^1 H(x)f(x)dx.
\end{aligned}$$

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