Problem 11854
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Proposed by R. Tauraso (Italy).

In the Euclidean plane, given a finite number of points $P_1, \ldots, P_n$, and a finite number of lines $l_1, \ldots, l_m$, prove that there is a half-line h such that for any point $Q \in h$, for any $k \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, $d(Q, l_k)$, the distance from Q to the line l_k, is less than $d(Q, P_j)$, the distance from Q to the point P_j.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

The set

$$S_{jk} := \left\{ Q \in \mathbb{R}^2 \ : \ d(Q, l_k) \geq d(Q, P_j) \right\}$$

is the closed interior of the parabola with focus P_j and directrix l_k (if $P_j \in l_k$ then S_{ij} degenerates to the line which intersect ortogonally l_k at P_j). Hence the problem is equivalent to showing that the set

$$\mathbb{R}^2 \setminus \bigcup_{(j,k)} S_{jk}$$

contains a half-line. Let r be a line which is not orthogonal to l_k for any $k \in \{1, \ldots, m\}$ (we can make such a choice because the set of lines is finite). Since the line r is not parallel to the axis of any parabola ∂S_{jk}, it follows that the intersection of r with S_{jk}, is a segment (possibly empty or just a single point). Therefore the line r certainly contains a half-line which is disjoint from the finite collection of those segments. $\square$