Problem 11853
(American Mathematical Monthly, Vol.122, August-September 2015)

Proposed by H. Ohtsuka (Japan).

Find

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n)} &= \sum_{n=1}^{\infty} \frac{2}{\exp(2^n) - \exp(-2^n)} \\ &= \sum_{n=1}^{\infty} \frac{2}{\exp(2^n)(1 - \exp(-2 \cdot 2^n))} \\ &= 2\sum_{n=1}^{\infty} \exp(-2^n) \sum_{k=0}^{\infty} \exp(-2 \cdot 2^n \cdot k) \\ &= 2\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \exp(-(2k+1) \cdot 2^n) \\ &= 2\sum_{m=1}^{\infty} e^{-2m} = \frac{2e^{-2}}{1 - e^{-2}} = \frac{2}{e^2 - 1},\end{aligned}$$

where we used that fact that, by the unique factorization theorem, any positive even integer $2m$ can be written in a unique way as $(2k+1) \cdot 2^n$ with $k \in \mathbb{N}$ and $n \in \mathbb{N}^+$. $\square$