Problem 11852
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For $n \in \mathbb{Z}^+$, let $\nu_n = k$ if 3^k divides n but 3^{k+1} does not. Let $x_1 = 2$, and for $n \geq 2$ let

$$x_n = 4\nu_n + 2 - \frac{2}{x_{n-1}}.$$

Show that every positive rational number appears exactly once in the sequence $\{x_n\}_{n\geq 1}$.

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The sequence begins with $2, 1, 4, \frac{3}{2}, \frac{2}{3}, 3, \frac{4}{3}, \frac{1}{2}, 6, \frac{5}{3}, \dots$. We have that for $n \geq 2$,

$$x_{3n} - 2 = 4\nu_{3n} - \frac{2}{x_{3n-1}} = 4(\nu_n + 1) - \frac{2}{2 - \frac{2}{x_{3n-2}}} = 4\nu_n + 4 - \frac{2}{2 - \frac{2}{2 - \frac{2}{x_{3n-3}}}} = 4\nu_n + 2 - \frac{2}{x_{3(n-1)} - 2}.$$

So $x_{3n} - 2$ satisfies the same recurrence as x_n. Since $x_3 - 2 = 2 = x_1$, it follows that for $n \geq 1$,

$$x_{3n} = x_n + 2.$$

Moreover, for $n \geq 1$,

$$x_{3n+1} = 2 - \frac{2}{x_{3n}} = 2 - \frac{2}{x_n + 2} \quad \text{and} \quad x_{3n+2} = 2 - \frac{2}{x_{3n+1}} = 1 - \frac{1}{x_n + 1}.$$

Now, we describe a procedure which determines, for any given rational number y_0, a unique positive integer n such that $x_n = y_0$.

1) Let $j = 0$ and let $c_0 = 0$.
2) If $y_j = 2$ then the value of n is equal to $c_j + 3^j$. Stop.
   If $y_j = 1$ then the value of n is equal to $c_j + 2 \cdot 3^j$. Stop.
3) If $2 < y_j$ then let $y_{j+1} = y_j - 2$ and let $c_{j+1} = c_j$.
   If $1 < y_j < 2$ then let $y_{j+1} = 2/(2 - y_j) - 2$ and let $c_{j+1} = c_j + 3^j$.
   If $0 < y_j < 1$ then let $y_{j+1} = 1/(1 - y_j) - 1$ and let $c_{j+1} = c_j + 2 \cdot 3^j$.
4) Add 1 to j and go to step 2).

Note that the rational number y_j is positive and if $y_j = a_j/b_j \notin \{1, 2\}$ where $a_j, b_j \in \mathbb{Z}^+$ then

$$y_{j+1} = \frac{a_{j+1}}{b_{j+1}} = \begin{cases} \frac{a_j - 2b_j}{b_j} & \text{if } 2 < y_j \quad \Rightarrow \quad a_{j+1} + b_{j+1} = a_j - b_j, \\ \frac{2a_j - 2b_j}{2b_j - a_j} & \text{if } 1 < y_j < 2 \quad \Rightarrow \quad a_{j+1} + b_{j+1} = a_j, \\ \frac{a_j}{b_j - a_j} & \text{if } 0 < y_j < 1 \quad \Rightarrow \quad a_{j+1} + b_{j+1} = b_j. \end{cases}$$

Hence $a_j + b_j > a_{j+1} + b_{j+1} \geq 1$ which implies that the procedure will eventually stop. $\square$