Problem 11851
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Proposed by D. M. Bătinețu-Giurgiu, and N. Stanciu (Romania).

For real a and b and integer $n \geq 1$, let $\gamma_n(a,b) = -\ln(n+a) + \sum_{k=1}^{n} \frac{1}{k+b}$.
(a) Prove that $\gamma(a,b) = \lim_{n\to\infty} \gamma_n(a,b)$ exists and is finite.
(b) Find

$$\lim_{n\to\infty} \left(\ln\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) \right)^n.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We have that if $x > 0$ then

$$\Psi(x) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x-1+k} \right) = \ln(x) - \frac{1}{2x} + O(1/x^2),$$

which implies that (b should be not a negative integer)

$$\begin{aligned}
\gamma_n(a,b) + \Psi(b+1) &= -\ln(n+a) + \sum_{k=1}^{n} \frac{1}{k+b} + \Psi(b+1) \\
&= -\ln(n+a) + \Psi(b+n+1) \\
&= -\ln(n+a) + \ln(b+n+1) - \frac{1}{2(b+n+1)} + O(1/n^2) \\
&= \ln\left(\frac{b+n+1}{n+a}\right) - \frac{1}{2(b+n+1)} + O(1/n^2) \\
&= \frac{b-a+\frac{1}{2}}{n} + O(1/n^2)
\end{aligned}$$

Therefore $\gamma(a,b) = -\Psi(b+1)$. Moreover,

$$\begin{aligned}
\ln\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) &= 1 - \ln(n+a) + \sum_{k=1}^{n} \frac{1}{k+b} + \Psi(b+1) \\
&= 1 + \frac{b-a+\frac{1}{2}}{n} + O(1/n^2).
\end{aligned}$$

and finally we get

$$\lim_{n\to\infty} \left(\ln\left(\frac{e}{n+a}\right) + \sum_{k=1}^{n} \frac{1}{k+b} - \gamma(a,b) \right)^n = \exp\left(b - a + \frac{1}{2}\right).$$

$\square$