Problem 11850
(American Mathematical Monthly, Vol.122, June-July 2015)

Proposed by Z. Ahmed (India).

Let A_n be the function given by

$$A_n(x) = \sqrt{\frac{2}{\pi}}\frac{1}{n!}(1+x^2)^{n/2}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right)$$

Prove that for nonnegative integers m and n,

$$\int_{-\infty}^{+\infty} A_m(x)A_n(x)dx = \delta(m,n),$$

where $\delta(m,n) = 1$ if $m = n$, and otherwise $\delta(m,n) = 0$.

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $I(m,n)$ be the desired integral. We first note that A_n has the same parity as n and $|A_n(x)| = O(1/x^2)$. So $I(m,n)$ is finite and, by symmetry, $I(m,n) = 0$ when $m+n$ is odd.
Now assume that $m+n$ is even. By the generalized to the Leibniz rule,

$$\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right) = \frac{d^n}{dx^n}\left(\frac{1}{(x+i)(x-i)}\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}\left(\frac{1}{(x-i)}\right)\cdot\frac{d^{n-k}}{dx^{n-k}}\left(\frac{1}{(x+i)}\right)$$
$$= (-1)^n n!\sum_{k=0}^{n}\frac{1}{(x-i)^{k+1}(x+i)^{n-k+1}}.$$

Hence, when $m+n$ is even, the integrand function is rational and we can use the residue theorem to evaluate the integral on the real line,

$$\begin{aligned}
I(m,n) &= \frac{2(-1)^{n+m}}{\pi}\int_{-\infty}^{+\infty}\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{(x-i)^{\frac{n+m}{2}}(x+i)^{\frac{n+m}{2}}}{(x+i)^{k+j+2}(x-i)^{n+m-k-j+2}}\,dx \\
&= 4i\sum_{k=0}^{n}\sum_{j=0}^{m}\text{Res}\left(\frac{(x-i)^{\frac{n+m}{2}}(x+i)^{\frac{n+m}{2}}}{(x-i)^{k+j+2}(x+i)^{n+m-k-j+2}}, i\right) \\
&= 4i\sum_{k=0}^{n}\sum_{j=0}^{m}[w^{-1}]\left(\frac{(w+2i)^{-\frac{n+m}{2}+k+j-2}}{w^{-\frac{n+m}{2}+k+j+2}}\right) \\
&= 4i\sum_{k=0}^{n}\sum_{j=0}^{m}[w^{-1}]\sum_{r\geq 0}\binom{-\frac{n+m}{2}+k+j-2}{r}\frac{(2i)^{-\frac{n+m}{2}+k+j-2-r}}{w^{-\frac{n+m}{2}+k+j+2-r}} \\
&= -\frac{1}{2}\sum_{r\geq 0}\binom{r-3}{r}P(n,m,r-1) \\
&= -\frac{1}{2}\left(\binom{-3}{0}P(n,m,-1) + \binom{-2}{1}P(n,m,0) + \binom{-1}{2}P(n,m,1)\right) \\
&= \begin{cases} -\frac{1}{2}((\min(m,n)+1) - 2(\min(m,n)+1) + (\min(m,n)+1)) = 0 & \text{if } m \neq n, \\ -\frac{1}{2}((m - 2(m+1) + m) = 1 & \text{if } m = n. \end{cases}
\end{aligned}$$

where $P(n,m,s)$ is the number of integer couples $(k,j) \in [0,n]\times[0,m]$ such that $k+j = \frac{n+m}{2}+s$.
□