Problem 11848
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Proposed by I. Mező (China).

Prove that

$$\frac{1}{2\pi}\mathrm{Li}_2(e^{-2\pi}) = \ln(2\pi) - 1 - \frac{5\pi}{12} - \sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{k(2k+1)}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy.

By the definition of Bernoulli numbers, we have that

$$F(x) := 2\sum_{k=1}^{\infty}(-1)^{k-1}\zeta(2k)x^{2k} = \sum_{k=1}^{n} B_{2k}\frac{(2\pi x)^{2k}}{(2k)!} = \frac{2\pi x}{e^{2\pi x}-1} - 1 + \pi x.$$

Hence

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{k(2k+1)} = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{2k} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{2k+1} = \int_0^1 \frac{F(x)}{x}dx - \int_0^1 F(x)dx.$$

As regards the first integral, we get

$$\int_0^1 \frac{F(x)}{x}dx = \pi + \int_0^{2\pi}\left(\frac{1}{e^t-1} - \frac{1}{t}\right)dt = \pi + \left[\ln(1-e^{-t}) - \ln(t)\right]_{0^+}^{2\pi} = \pi + \ln(1-e^{-2\pi}) - \ln(2\pi).$$

The second one goes as follows

$$\int_0^1 F(x)dx = -1 + \frac{\pi}{2} + \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{e^t-1}dt = -1 + \frac{\pi}{2} - \frac{1}{2\pi}\mathrm{Li}_2(e^{-2\pi}) + \ln(1-e^{-2\pi}) + \frac{\pi}{12}.$$

Infact

$$\begin{aligned}\int_0^{2\pi}\frac{t}{e^t-1}dt &= \int_0^{2\pi}\frac{te^{-t}}{1-e^{-t}}dt = \int_0^{2\pi} t\sum_{k=1}^{\infty}e^{-kt}dt = \sum_{k=1}^{\infty}\int_0^{2\pi} te^{-kt}dt \\ &= -\sum_{k=1}^{\infty}\left[\frac{e^{-kt}}{k^2} + \frac{te^{-kt}}{k}\right]_0^{2\pi} = -\mathrm{Li}_2(e^{-2\pi}) + 2\pi\ln(1-e^{-2\pi}) + \frac{\pi^2}{6}.\end{aligned}$$

Finally

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{k(2k+1)} &= \left(\pi + \ln(1-e^{-2\pi}) - \ln(2\pi)\right) - \left(-1 + \frac{\pi}{2} - \frac{1}{2\pi}\mathrm{Li}_2(e^{-2\pi}) + \ln(1-e^{-2\pi}) + \frac{\pi}{12}\right) \\ &= \frac{1}{2\pi}\mathrm{Li}_2(e^{-2\pi}) - \ln(2\pi) + 1 + \frac{5\pi}{12}.\end{aligned}$$

□