Problem 11847
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Proposed by M. Bencze (Romania).

Prove that for $n \geq 1$,

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\ln^2(1+1/k)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

For $x \in (0,1]$, we have that

$$\frac{2x}{2+x} < \ln(1+x) < \frac{x}{\sqrt{1+x}}.$$

Infact

$$f(x) := \ln(1+x) - \frac{2x}{2+x}, \quad f'(x) = \frac{x^2}{(1+x)(2+x)^2}, \quad f''(x) = \frac{x(4+2x-x^2)}{(1+x)^2(2+x)^3},$$

which implies that $f(x)$ is strictly convex in $(0,1]$, and $f(x) > f(0) + f'(0)x = 0$ in $(0,1]$. Moreover

$$g(x) := \frac{x}{\sqrt{1+x}} - \ln(1+x), \quad g'(x) = \frac{x+2-2\sqrt{1+x}}{2(1+x)^{3/2}}, \quad g''(x) = \frac{4\sqrt{1+x}-x-4}{4(1+x)^{5/2}},$$

which implies that $g(x)$ is strictly convex in $(0,1]$, and $g(x) > g(0) + g'(0)x = 0$ in $(0,1]$.
Hence, for $1 \leq k \leq n$,

$$k(k+1) = \frac{1+1/k}{(1/k)^2} < \frac{1}{\ln^2(1+1/k)} < \left(\frac{2+1/k}{2/k}\right)^2 = k(k+1) + \frac{1}{4}$$

and adding up we get

$$\frac{n(n+1)(n+2)}{3} = \sum_{k=1}^{n} k(k+1) < \sum_{k=1}^{n} \frac{1}{\ln^2(1+1/k)} < \sum_{k=1}^{n} \left(k(k+1) + \frac{1}{4}\right) = \frac{n(n+1)(n+2)}{3} + \frac{n}{4}.$$

$\square$