Problem 11844
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For nonnegative integers m and n, prove

$$\sum_{k=0}^{n}(m-2k)\binom{m}{k}^3 = (m-n)\binom{m}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We prove by induction with respect to n that

$$\sum_{k=0}^{n}(m-2k)\binom{m}{k}^3 = (m-n)\binom{m}{n}T(m-1,n)$$

where $T(M,N) = \sum_{j=0}^{M}\binom{j}{N}\binom{j}{M-N} = T(M,M-N)$. For $n=0$,

$$\sum_{k=0}^{0}(m-2k)\binom{m}{k}^3 = m = mT(m-1,0).$$

As regards the inductive step, it holds as soon as

$$\begin{aligned}(m-2n)\binom{m}{n}^3 &= (m-n)\binom{m}{n}T(m-1,n) - (m-(n-1))\binom{m}{n-1}T(m-1,n-1)\\ &= (m-n)\binom{m}{n}T(m-1,n) - n\binom{m}{n}T(m-1,n-1)\end{aligned}$$

which is equivalent to

$$(m-2n)\binom{m}{n}^2 = (m-n)T(m-1,n) - nT(m-1,n-1). \tag{1}$$

Now we show (1).

$$\begin{aligned}(m-n)T(m,n) &= (m-n)\sum_{j=0}^{m}\binom{j}{n}\binom{j}{m-n}\\ &= (m-n)\sum_{j=0}^{m-1}\binom{j}{n}\frac{j-(m-n)+1}{m-n}\binom{j}{m-n-1} + (m-n)\binom{m}{n}^2\\ &= -(m-n)T(m-1,n) + \sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1}(j+1) + (m-n)\binom{m}{n}^2.\end{aligned}$$

In a similar way

$$\begin{aligned}
nT(m,n) &= n\sum_{j=0}^{m}\binom{j}{n}\binom{j}{m-n}\\
&= n\sum_{j=0}^{m-1}\binom{j}{n}\frac{j-n+1}{n}\binom{j}{m-n}+n\binom{m}{n}^2\\
&= -nT(m-1,n-1)+\sum_{j=0}^{m-1}\binom{j}{n-1}\binom{j}{m-n}(j+1)+n\binom{m}{n}^2.
\end{aligned}$$

By subtracting this equation from the previous one, we get

$$\begin{aligned}
(m-2n)T(m,n) &= -(m-n)T(m-1,n)+nT(m-1,n-1)+(m-2n)\binom{m}{n}^2\\
&\quad+\sum_{j=0}^{m-1}\left(\binom{j}{n}\binom{j}{m-n-1}-\binom{j}{n-1}\binom{j}{m-n}\right)(j+1)\\
&= -(m-n)T(m-1,n)+nT(m-1,n-1)+(m-2n)\binom{m}{n}^2\\
&\quad+(m-2n)\sum_{j=0}^{m-1}\binom{j+1}{n}\binom{j+1}{m-n}\\
&= -(m-n)T(m-1,n)+nT(m-1,n-1)+(m-2n)\binom{m}{n}^2\\
&\quad+(m-2n)T(m,n),
\end{aligned}$$

that is

$$0=-(m-n)T(m-1,n)+nT(m-1,n-1)+(m-2n)\binom{m}{n}^2.$$

□