Problem 11843
(American Mathematical Monthly, Vol.122, May 2015)

Proposed by M. Bencze (Romania).

Let n and k be positive integers, and let $x_j \geq 1$ for $1 \leq j \leq n$. Let $y = \prod_{i=1}^n x_i$. Show that

$$\sum_{i=1}^n \frac{1}{1+x_i} \geq \sum_{j=1}^n \frac{1}{1+(x_j^{k-1}y)^{1/(n+k-1)}}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Let $f(x) = 1/(1+e^x)$ then

$$f''(x) = \frac{e^x\,(e^x-1)}{(1+e^x)^3}$$

which implies that f is convex in $[0,+\infty)$. Since $\ln(x_j) \geq 0$ for $1 \leq j \leq n$, it follows that

$$\frac{\sum_{i=1}^n f(\ln(x_i)) + (k-1)f(\ln(x_j))}{n+k-1} \geq f\left(\frac{\sum_{i=1}^n \ln(x_i) + (k-1)\ln(x_j)}{n+k-1}\right),$$

that is

$$\frac{1}{n+k-1}\left(\sum_{i=1}^n \frac{1}{1+x_i} + \frac{k-1}{1+x_j}\right) \geq \frac{1}{1+(x_j^{k-1}y)^{1/(n+k-1)}}.$$

Hence, by summing over j, we obtain

$$\frac{1}{n+k-1}\sum_{j=1}^n\left(\sum_{i=1}^n \frac{1}{1+x_i} + \frac{k-1}{1+x_j}\right) \geq \sum_{j=1}^n \frac{1}{1+(x_j^{k-1}y)^{1/(n+k-1)}},$$

which yields the required inequality as soon as we note that

$$\sum_{j=1}^n\left(\sum_{i=1}^n \frac{1}{1+x_i} + \frac{k-1}{1+x_j}\right) = n\sum_{i=1}^n \frac{1}{1+x_i} + (k-1)\sum_{j=1}^n \frac{1}{1+x_j} = (n+k-1)\sum_{i=1}^n \frac{1}{1+x_i}.$$

□