Problem 11842
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Proposed by I. Mező (China).

Let ψ be the Digamma function, that is, $\psi(x) = (\log\Gamma(x))'$. Let $\phi = (1+\sqrt{5})/2$. Prove that

$$\sum_{n=1}^{\infty} \frac{\psi(n+\phi)-\psi(n-1/\phi)}{n^2+n-1} = \frac{\pi^2}{2\sqrt{5}} + \frac{\pi^2\tan^2(\sqrt{5}\pi/2)}{\sqrt{5}} + \frac{4\pi\tan(\sqrt{5}\pi/2)}{5}.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

Since $\psi(n+z) = \psi(a) + \sum_{k=0}^{n-1}\frac{1}{k+z}$, it follows that if a and b are not non-positive integers then for all integers $N \geq 0$,

$$\begin{aligned}\sum_{n=0}^{N} \frac{\psi(n+a)-\psi(n+b)}{(n+a)(n+b)} =&(\psi(a)-\psi(b))\sum_{n=0}^{N}\frac{1}{(n+a)(n+b)}\\ &+\frac{(b-a)}{2}\left(\left(\sum_{n=0}^{N}\frac{1}{(n+a)(n+b)}\right)^2-\sum_{n=0}^{N}\frac{1}{((n+a)(n+b))^2}\right).\end{aligned}$$

In our case $a=\phi$ and $b=-1/\phi$, and as N goes to infinity we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\psi(n+\phi)-\psi(n-1/\phi)}{n^2+n-1} =&(\psi(\phi)-\psi(-1/\phi))\left(1+\sum_{n=0}^{\infty}\frac{1}{n^2+n-1}\right)\\ &-\frac{\sqrt{5}}{2}\left(\left(\sum_{n=0}^{\infty}\frac{1}{n^2+n-1}\right)^2-\sum_{n=0}^{\infty}\frac{1}{(n^2+n-1)^2}\right).\end{aligned}$$

The required formula is proved as soon as we compute $\psi(\phi)-\psi(-1/\phi)$ and the two series.

i) By the reflection formula,

$$\psi(\phi)-\psi(-1/\phi) = \psi(\phi)-\psi(1-\phi) = -\frac{\pi}{\tan(\pi\phi)} = \pi\tan(\sqrt{5}\pi/2).$$

ii) By the residue formula, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n^2+n-1} &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{n^2+n-1} = \frac{\pi}{2}\sum_{n\in\mathbb{Z}}\text{Res}\left(\frac{\cot(\pi z)}{z^2+z-1}, z\right)\\ &= -\frac{\pi}{2}\left(\text{Res}\left(\frac{\cot(\pi z)}{z^2+z-1}, -\phi\right)+\text{Res}\left(\frac{\cot(\pi z)}{z^2+z-1}, 1/\phi\right)\right)\\ &= -\frac{\pi\cot(\pi\phi)}{\sqrt{5}} = \frac{\pi\tan(\sqrt{5}\pi/2)}{\sqrt{5}}.\end{aligned}$$

ii) By the residue formula, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{(n^2+n-1)^2} &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{(n^2+n-1)^2} = \frac{\pi}{2}\sum_{n\in\mathbb{Z}}\text{Res}\left(\frac{\cot(\pi z)}{(z^2+z-1)^2}, z\right)\\ &= -\frac{\pi}{2}\left(\text{Res}\left(\frac{\cot(\pi z)}{(z^2+z-1)^2}, -\phi\right)+\text{Res}\left(\frac{\cot(\pi z)}{(z^2+z-1)^2}, 1/\phi\right)\right)\\ &= -\frac{\pi}{5}\left(-\frac{\pi}{\sin^2(\pi\phi)}-\frac{2\cot(\pi\phi)}{\sqrt{5}}\right) = \frac{\pi^2}{5}+\frac{\pi^2\tan^2(\sqrt{5}\pi/2)}{5}-\frac{2\pi\tan(\sqrt{5}\pi/2)}{5\sqrt{5}}.\end{aligned}$$

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