Problem 11841
(American Mathematical Monthly, Vol.122, May 2015)

Proposed by L. Giugiuc (Romania).

Let $ABCD$ be a convex quadrilateral. Let E be the midpoint of AC, and let F be the midpoint of BD. Show that

$$|AB| + |BC| + |CD| + |DA| \geq |AC| + |BD| + 2|EF|.$$

Solution proposed by Roberto Tauraso, Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica, 00133 Roma, Italy.

We show that the inequality holds for any quadrilateral (even if it is not convex).
Let $A, B, C, D \in \mathbb{C}$ then $2E = A + C$, $2F = B + D$, and we have to show that

$$|A - B| + |B - C| + |C - D| + |D - A| \geq |A - C| + |B - D| + |A + C - B - D|,$$

which is Hlawka's inequality

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |x + z|$$

with $x = A - B$, $y = B - C$, and $z = C - D$.
A short proof of Hlawka's inequality is the following. After straightforward calculations we obtain

$$\begin{aligned}
&(|x| + |y| + |z| + |x + y + z|)(|x| + |y| + |z| + |x + y + z| - |x + y| - |y + z| - |x + z|) \\
&= (|x| + |y| - |x + y|)(|z| + |x + y + z| - |x + y|) + (|y| + |z| - |y + z|)(|x| + |x + y + z| - |y + z|) \\
&\quad + (|z| + |x| - |x + z|)(|y| + |x + y + z| - |x + z|) + |x|^2 + |y|^2 + |z|^2 + |x + y + z|^2 \\
&- |x + y|^2 - |y + z|^2 - |x + z|^2 \geq 0
\end{aligned}$$

which holds by triangle inequality and the fact that

$$|x|^2 + |y|^2 + |z|^2 + |x + y + z|^2 = |x + y|^2 + |y + z|^2 + |x + z|^2$$

is verified for all complex numbers x, y, z. □